

Noncommutative Iwasawa theory arising from Hecke algebras

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Abstract

Let p be an odd prime and f be a nearly ordinary Hilbert modular Hecke eigenform defined over a totally real field F . Let \mathbb{I} be an irreducible component of the universal nearly ordinary or locally cyclotomic deformation of the representation of Gal_F that is associated to f . We study the deformation rings over a p -adic Lie extension F_∞ that contains the cyclotomic \mathbb{Z}_p -extension of F . More precisely, we prove a control theorem about these rings. We introduce a category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$, where $\mathcal{G} = \mathrm{Gal}(F_\infty/F)$ and $\mathcal{H} = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$, which is the category of modules which are torsion with respect to a certain Ore set, which generalizes the Ore set introduced by Venjakob. For Selmer groups which are in this category, we formulate a Main conjecture in the spirit of Noncommutative Iwasawa theory. We then set up a strategy to prove the conjecture by generalizing work of Burns, Kato, Kakde, and Ritter and Weiss. This requires appropriate generalizations of results of Oliver and Taylor, and Oliver on Logarithms of certain K -groups, which we have presented here.

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 2 |
| 2 | Nearly ordinary Hilbert modular Hecke algebra | 5 |
| 2.1 | Adelic Hilbert Modular forms | 5 |
| 2.2 | Hecke algebras | 8 |
| 2.3 | Locally cyclotomic Hecke algebra | 9 |
| 2.4 | Modular Galois representations | 9 |
| 2.5 | Locally cyclotomic deformation | 10 |
| 3 | Deformation rings and base change | 11 |
| 3.1 | Base change of deformation rings | 11 |
| 3.2 | Control of deformation rings | 12 |
| 4 | Adjoint Selmer groups and Kähler differentials | 12 |
| 4.1 | Selmer groups | 12 |
| 4.2 | Kähler differentials | 13 |
| 4.3 | Control of Kähler differentials | 14 |
| 5 | Deformation rings in an <i>admissible</i> tower E_∞/E | 15 |
| 5.1 | Admissible p -adic Lie extension | 15 |
| 5.2 | Example over decomposition groups | 16 |
| 6 | Noncommutative Iwasawa theory of $\mathrm{Sel}_{E_\infty}^*(\mathrm{Ad}^0(\phi))$ | 18 |
| 6.1 | Ore sets and the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$ | 18 |
| 6.2 | Noetherian Deformation rings and $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$ | 21 |
| 6.3 | Noetherian property of \mathcal{R}_∞ | 23 |

| | | |
|----------|---|-----------|
| 6.4 | Periods of adjoint Galois representations | 23 |
| 6.5 | Non-commutative Main conjecture | 24 |
| 6.6 | Main conjecture in the abelian case | 25 |
| 6.7 | Remark on zeros | 26 |
| 7 | K_1 computations and congruences over $\mathbb{I}[[\mathcal{G}]]$ | 27 |
| 7.1 | General strategy | 27 |
| 7.2 | The Logarithm map over $\widehat{\mathbb{I}(\Gamma^{p^e})}$ | 33 |
| 7.3 | The Integral logarithm over $\mathbb{I}[[\mathcal{G}]]$ | 36 |
| 7.4 | The Logarithm map over $\widehat{\mathbb{I}(Z)}_{(p)}$ | 42 |
| 7.5 | The Integral Logarithm over $\widehat{\mathbb{I}(Z)}_{(p)}$ | 43 |
| 7.6 | The Logarithm map under restriction maps | 44 |
| 7.7 | Congruences over $\mathbb{I}[[\mathcal{G}]]$ | 47 |
| 8 | Relations between the congruences over $\mathbb{I}[[\mathcal{G}]]$ and $\mathbb{Z}_p[[\mathcal{G}]]$ | 52 |
| 8.1 | Congruences over $\mathbb{Z}_p[[\mathcal{G}]]$ | 52 |
| 8.2 | Relation between the congruences | 53 |
| 9 | Application to p-adic L-function | 54 |
| 9.1 | Torsion Congruences and p -adic L-function | 54 |
| 9.2 | Remarks on Torsion Congruence in Families | 57 |

1 Introduction

Let F be a totally real number field and F_∞ be a Galois extension such that $\text{Gal}(F_\infty/F)$ is a pro- p , p -adic Lie group \mathcal{G} for an odd prime p . We assume that the cyclotomic \mathbb{Z}_p -extension F_{cyc} is contained in F_∞ . For every $n \in \mathbb{N}$, let F_n be a finite Galois extension of F contained in F_∞ such that $F_n \subset F_{n+1}$ and $F_\infty = \cup_n F_n$.

Over the totally real field F , consider a Hecke eigenform $f_0 \in S_\kappa^{n,ord}(\mathfrak{N}, \varepsilon_0; W)$ of weight $\kappa = (0, I)$ (see §2.1 for a precise definition of these weights). Let ρ_0 be the representation of $\text{Gal}(\overline{F}/F)$ that is associated to f_0 . As $\text{Gal}(F_\infty/F)$ is a pro- p , p -adic Lie group it is solvable and we can consider the base change f_n of f_0 to the totally real field F_n . Then the representation $\rho_n := \rho_{f_n}$ is isomorphic to the restriction $\rho_n := \rho_0|_{\text{Gal}_{F_n}}$. For each of these representations, we consider the deformations of the residual representations $\overline{\rho}_n$. Assuming the conditions **(sf)**, **(h1)**-**(h4)** which we present in Section 2.5 and also absolute irreducibility of the residual representation $\overline{\rho}_0$, a universal deformation ring \mathcal{R}_{F_n} exists for $\overline{\rho}_n$ for all n . In this article, we study the deformation rings over the p -adic Lie extension F_∞ . This allows us to study the Selmer groups of the adjoint representations $\text{Ad}^0(\rho_0)$ defined over the p -adic Lie extension F_∞ . In fact, if ρ_0 denotes the deformation of ρ_0 , then we consider the Selmer groups of the adjoint $\text{Ad}^0(\rho_0)$ along an irreducible component \mathbb{I} of \mathcal{R}_0 . Let $\mathcal{G} := \text{Gal}(F_\infty/F)$ and $\mathcal{H} := \text{Gal}(F_\infty/F_{cyc})$. Then there is a natural action of the group \mathcal{G} on these Selmer groups, making these Selmer groups modules over $\mathbb{I}[[\mathcal{G}]]$ in a natural way.

We formulate a Main conjecture for Selmer groups of $\text{Ad}^0(\rho_0)$ along the irreducible component \mathbb{I} . A Main conjecture for Galois representations attached to the ordinary Hida family was also formulated by Barth in his thesis [Bar09]. Our formulation is slightly different from his formulation. A noncommutative generalization of the Main Conjecture of Iwasawa theory was presented in the paper [CFKSV05] for Galois representations arising from motives which are ordinary at a prime p . A requirement for this formulation is that the Pontryagin dual of Selmer groups defined over F_∞ are in the category $\mathfrak{M}_{\mathcal{H}}(\mathcal{G})$. This category $\mathfrak{M}_{\mathcal{H}}(\mathcal{G})$ consists of finitely generated modules over $\mathbb{Z}_p[[\mathcal{G}]]$ and torsion with respect to a certain Ore Set (see Section 7). This Ore set is defined to be the set of all the elements of $\mathbb{Z}_p[[\mathcal{G}]]$ such that $\mathbb{Z}_p[[\mathcal{G}]]/x$ is a finitely generated module over $\mathbb{Z}_p[[\mathcal{H}]]$. It is conjectured in [CFKSV05] that the Selmer groups defined over F_∞ attached to p -ordinary Galois representations are in the category $\mathfrak{M}_{\mathcal{H}}(\mathcal{G})$.

As a generalization, we consider the set \mathcal{S} which consists of elements $x \in \mathbb{I}[[\mathcal{G}]]$ such that $\mathbb{I}[[\mathcal{G}]]/x$ is a finitely generated module over $\mathbb{I}[[\mathcal{H}]]$. Then we consider the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}[[\mathcal{G}]]$ which consists of finitely generated modules over $\mathbb{I}[[\mathcal{G}]]$ which are \mathcal{S} -torsion (see section 6 for more details). To formulate a noncommutative Main conjecture, we also require that the Selmer group of $\text{Ad}^0(\rho_0)$ along \mathbb{I} is in the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$. We state this as a conjecture and in fact, this is a straight forward generalization of the conjecture for the Selmer group of $\text{Ad}^0(\rho_0)$ ([CFKSV05]). This conjecture still remains hard to understand even in the in the first non-trivial and crucial case, namely the case when \mathcal{G} is a 2 dimensional p -adic Lie group. In this case, when the Selmer group for $\text{Ad}^0(\rho_0)$ has its μ -invariant defined over the cyclotomic \mathbb{Z}_p -extension is zero, Theorem B, below seems to indicate the difficulty in trying to solve this problem as \mathcal{R}_{∞} is not noetherian. We confess that we began with the hope that we may be able to say something about \mathcal{R}_{∞} .

The main conjecture for Selmer group of $\text{Ad}^0(\rho_0)$ along \mathbb{I} when F_{∞} is the cyclotomic \mathbb{Z}_p -extension of F was studied by Hida in many papers. However, for most of the parts in this article we refer to the book [Hid06].

Our aim in this paper is to explore the noncommutative Iwasawa theory for a p -adic family of modular forms and also take Hida's results along a new direction. After formulating the Main conjecture for Selmer groups over $\mathbb{I}[[\mathcal{G}]]$, we also show that a suitable generalization of the strategy of Burns, Kato, Kakde and Hara can be used to prove the Main conjecture over $\mathbb{I}[[\mathcal{G}]]$. Their strategy has been successfully used to prove the main conjecture over totally real fields and for the trivial Galois representation. We also give generalizations of the torsion congruences which played a crucial role in the proof of the noncommutative Main conjecture over totally real fields.

We first show a relation between the conjectures regarding the categories $\mathfrak{M}_{\mathcal{H}}(\mathcal{G})$ and $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$.

Theorem A (Theorem 6.16). Consider the representation $\rho_{\mathbb{I}} : \text{Gal}_F \rightarrow GL_2(\mathbb{I})$ which arises from the irreducible component \mathbb{I} and let $\phi_k : \mathbb{I} \rightarrow \mathcal{O}$ be a morphism of local algebras which give rise to a locally cyclotomic point P of weight k . The dual Selmer group $\text{Sel}_{E_{\infty}}^*(\text{Ad}^0(\rho_{\mathbb{I}}))$ is \mathcal{S} -torsion if and only if $\text{Sel}_{E_{\infty}}^*(\text{Ad}^0(\rho_P))$ is S -torsion.

Theorem B (Prop 6.17). Let F_{∞} be a p -adic Lie extension of a totally real field F such that $\mathcal{G} := \text{Gal}(F_{\infty}/F)$ is a p -adic Lie group of dimension two. Let $\mathcal{H} := \text{Gal}(F_{\infty}/F_{\text{cyc}})$ and $\Gamma := \text{Gal}(F_{\text{cyc}}/F)$. Then

- (i) the dual Selmer group $\Omega_{\mathcal{R}_{\infty}/W} \otimes W$ is a finitely generated module over $W[[\mathcal{H}]]$,
- (ii) the ring \mathcal{R}_{∞} is not noetherian.

Over the cyclotomic \mathbb{Z}_p -extension, results of Hida show that the noetherian property of \mathcal{R}_{cyc} is related to the vanishing of μ -invariant of the dual Selmer group of $\text{Ad}^0(\rho)$. However, as a deviation from the cyclotomic theory, we come across the strange property that the ring \mathcal{R}_{∞} is not noetherian.

Let $\mathbb{I} \cong \mathcal{O}[[X_1, \dots, X_r]]$, for some r , with \mathcal{O} unramified over \mathbb{Z}_p . and \mathcal{G} a p -adic Lie group of dimension 1. Let $\Sigma(\mathcal{G})$ be any set of rank 1 subquotients of \mathcal{G} of the form U^{ab} with U an open subgroup of \mathcal{G} that has the following property:

- (*) For each Artin representation ρ of \mathcal{G} , there is a finite subset $\{U_i^{ab} : i \in I\}$ of $\Sigma(\mathcal{G})$ and for each index i an integer m_i and a degree one representation ρ_i of U_i^{ab} such that there is an isomorphism of virtual representations $\rho \cong \sum_{i \in I} m_i \cdot \text{Ind}_{U_i}^{\mathcal{G}} \text{Ind}_{U_i^{ab}}^{U_i} \rho_i$.

Let U^{ab} be a subquotient satisfying the above property (*), and for any group G , let $\mathbb{I}(G) := \mathbb{I}[[G]]$. Note that we have the following natural homomorphism,

$$(1) \quad K_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \rightarrow K_1(\mathbb{I}(U)_{\mathcal{S}}) \rightarrow K_1(\mathbb{I}(U^{ab})_{\mathcal{S}}) \rightarrow \mathbb{I}(U^{ab})_{\mathcal{S}}^{\times} \subset Q_{\mathbb{I}}(U^{ab})^{\times}.$$

Taking all the U^{ab} in $\Sigma(\mathcal{G})$ we get the following homomorphism

$$(2) \quad \Theta_{\Sigma(\mathcal{G})} : K_1(\mathbb{I}[[\mathcal{G}]]) \rightarrow \prod_{U^{ab} \in \Sigma(\mathcal{G})} Q_{\mathbb{I}}(U^{ab})^{\times}.$$

For any subgroup P of $\overline{\mathcal{G}}$, we write $\Theta_P^{\overline{\mathcal{G}}, ab}$ for the following natural composite homomorphism

$$K'_1(\mathbb{I}[[\mathcal{G}]]) \xrightarrow{\Theta_P^{\overline{\mathcal{G}}}} K_1(\mathbb{I}(U_P)) \longrightarrow K_1(\mathbb{I}(U_P^{ab})) \cong \mathbb{I}(U_P^{ab})^\times,$$

where the isomorphism is induced by taking determinants over $\mathbb{I}(U_P^{ab})$.

Theorem C (Theorem 7.40). Let $\Xi \in K'_1(\mathbb{I}[[\mathcal{G}]])$ and for all subgroups P of $\overline{\mathcal{G}}$, put $\Xi_P := \Theta_P^{\overline{\mathcal{G}}, ab}(\Xi) \in \mathbb{I}(U_P^{ab})^\times$.

(i) For all subgroups P, P' of $\overline{\mathcal{G}}$ with $[P', P'] \leq P \leq P'$, we have

$$\mathrm{Nr}_P^{P'}(\Xi_{U_{P'}^{ab}}) = \Pi_P^{P'}(\Xi_{U_P^{ab}}).$$

(ii) For all subgroups P of $\overline{\mathcal{G}}$ and all g in $\overline{\mathcal{G}}$ we have $\Xi_{gU_P^{ab}g^{-1}} = g\Xi_{U_P^{ab}}g^{-1}$.

(iii) For every $P \in \overline{\mathcal{G}}$ and $P \neq (1)$, we have

$$\mathrm{ver}_P^{P'}(\Xi_{U_{P'}^{ab}}) \equiv \Xi_{U_P^{ab}} \pmod{\mathcal{T}_{P, P'}} \text{ (resp. } \mathcal{T}_{P, P', \mathcal{S}} \text{ and } \widehat{\mathcal{T}}_{P, P'}).$$

(iv) For all $P \in C(\overline{\mathcal{G}})$ we have $\alpha_P(\Xi_{U_P^{ab}}) \equiv \prod_{P' \in C_P(\overline{\mathcal{G}})} \alpha_{P'}(\Xi_{U_{P'}^{ab}}) \pmod{p\mathcal{T}_P}$.

Conversely, if $\Xi_{U_P^{ab}} \in \mathbb{I}(U_P^{ab})^\times$ for all subgroups P of $\overline{\mathcal{G}}$, such that the above congruences hold then there exists an element $\Xi \in K'_1(\mathbb{I}(\mathcal{G}))$ such that $\Theta_P^{\overline{\mathcal{G}}, ab}(\Xi) = \Xi_{U_P^{ab}} \in \mathbb{I}(U_P^{ab})^\times$.

Crucial in the proof is the existence of the following logarithmic map $K'_1(\mathbb{I}[[\mathcal{G}]]) \xrightarrow{\mathfrak{L}} \mathbb{I}(Z)[\mathrm{Conj}(\overline{\mathcal{G}})]^\tau$. Further, we show that the integral logarithm map fits in the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu(\mathcal{O}) \times \mathbb{W} \times \mathcal{G}^{ab} & \longrightarrow & K'_1(\mathbb{I}[[\mathcal{G}]]) & \xrightarrow{\mathfrak{L}} & \mathbb{I}(Z)[\mathrm{Conj}(\overline{\mathcal{G}})]^\tau \longrightarrow \mathbb{W} \times \mathcal{G}^{ab} \longrightarrow 1 \\ & & \downarrow = & & \downarrow \Theta^{\overline{\mathcal{G}}} & & \downarrow \cong \beta^{\overline{\mathcal{G}}} \\ 1 & \longrightarrow & \mu(\mathcal{O}) \times \mathbb{W} \times \mathcal{G}^{ab} & \longrightarrow & \Phi^{\overline{\mathcal{G}}} & \xrightarrow{\mathcal{L}} & \Psi^{\overline{\mathcal{G}}} \longrightarrow \mathbb{W} \times \mathcal{G}^{ab} \longrightarrow 1, \end{array}$$

where $\mathbb{W} := (1 + p\mathbb{Z}_p)^r$, and \mathfrak{L} and \mathcal{L} are the integral logarithm maps. In Theorem 7.47, we show that the map $\Theta^{\overline{\mathcal{G}}}$ is an isomorphism and the congruences in the theorem above are derived from this isomorphism.

Theorem D (Theorem 7.47). The map $\Theta^{\overline{\mathcal{G}}}$ is an isomorphism.

To mention briefly, the logarithm maps transfer the multiplicative theory to the additive theory. To achieve this, among many other algebraic results, we need a generalization of a classical result of Higman in [Hig40], regarding the torsion subgroup of units of group rings. More precisely, we have the following generalization of Higman's theorem:

Theorem E (Theorem 7.23). For any finite p -group G , we have

$$(K_1(\mathbb{I}[G]))_{\mathrm{tors}} \cong \mu_K \times G^{ab} \times SK_1(\mathbb{I}[G]).$$

In Section 2, we recall Hilbert modular forms, the action of the Hecke algebra on the space of nearly ordinary Hilbert modular forms and the nearly ordinary Hecke algebra. In section 3, we recall the deformation of two-dimensional representations of the Galois group Gal_F , where F is a totally real field. We then recall the results that relate the nearly ordinary deformation to the nearly ordinary Hilbert modular forms both in Sections 2 and 3, in some detail, keeping in mind future applications and also for the convenience of the reader. In Section 4 we introduce the Selmer groups of the adjoint representation and then prove a control theorem. We recall that Selmer groups can be viewed as Kähler differentials. We then study the deformation rings over a p -adic Lie extension in section 5. In Section 6, we give a sufficient condition for Selmer

groups to be in the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$ in terms of the deformation rings. We then present a noncommutative Main Conjecture for these Selmer groups.

We also give some results regarding the structure of the deformation rings over the p -adic Lie extension F_{∞} . In Section 7, we extend the strategy of Burns, Kato, Kakde, and of Ritter and Weiss to prove the Main Conjecture. This section has some results on K -groups and logarithm maps that may be of independent interest. We extend some of the results of Oliver and this required us to generalize results on logarithm maps and computation of certain K -groups. In addition, we have defined a suitable generalization of the SK_1 -groups in Definition 7.20. The results in this section may be used to establish the Main Conjecture of a p -adic family of Galois representations arising from motives. In the section after this, we show that the p -adic L-function over $\mathbb{I}[[\mathcal{G}]]$ specializes to the p -adic L-function for each of the members in the family.

Section 2, where we have recalled the main results regarding Hecke algebras is very crucial for our work. It is evident how our results here are a generalization of results of Hida. Along with this, section 7 is the main section where we do the computation of the K -groups and it is clear how many of our results are generalizations of those of Burns, Kato, Kakde, Ritter and Weiss, and Oliver. This section owes a lot to their works. In addition, the paper [CFKSV05] of Coates, Fukaya, Kato, Sujatha and Venjakob has been a strong influence on our work.

2 Nearly ordinary Hilbert modular Hecke algebra

2.1 Adelic Hilbert Modular forms

Let F be a totally real number field, and \mathcal{O} denote the ring of integers of F . Let \mathfrak{N} denote an integral ideal of F . Consider the algebraic group $G = \text{Res}_{\mathcal{O}/\mathbb{Z}} GL(2)$ over \mathbb{Z} . Then for each commutative ring A , we have $G(A) = GL_2(A \otimes_{\mathbb{Z}} \mathcal{O})$. Let $T_0 = \mathbb{G}_{m/\mathcal{O}}^2$ be the diagonal torus of $GL(2)_{/\mathcal{O}}$. Then consider $T = \text{Res}_{\mathcal{O}/\mathbb{Z}}$ and $T_G = \text{Res}_{\mathcal{O}/\mathbb{Z}} T_0$. Then T_G contains the center Z of G .

Writing $I = \text{Hom}_{\text{field}}(F, \overline{\mathbb{Q}})$, the group of algebraic characters $X(T_G) = \text{Hom}_{\text{alg gp}}(T_G/\overline{\mathbb{Q}}, \mathbb{G}_{m/\overline{\mathbb{Q}}})$ can be identified with $\mathbb{Z}[I]^2$ so that $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$ induces the following character on $T_G(\mathbb{Q}) = F^{\times} \times F^{\times}$:

$$T_G(\mathbb{Q}) \longrightarrow \overline{\mathbb{Q}}^{\times} : (\xi_1, \xi_2) \mapsto \kappa(\xi_1, \xi_2) = \xi_1^{\kappa_1} \xi_2^{\kappa_2},$$

where $\xi_j^{\kappa_j} = \prod_{\sigma \in I} \sigma(\xi_j)^{\kappa_{j,\sigma}} \in \overline{\mathbb{Q}}^{\times}$. Then consider the ‘‘Neben’’ characters defined as the triple

$$\varepsilon = (\varepsilon_1, \varepsilon_2 : T(\widehat{\mathbb{Z}}) \longrightarrow \mathbb{C}^{\times}, \varepsilon_+ : Z(\mathbb{A})/Z(\mathbb{Q}) \longrightarrow \mathbb{C}^{\times}).$$

This is the way in which the Neben characters have been considered in [Hid06, 2.3.2] and this is so defined so that the character ε_+ is the central character of the automorphic form that corresponds to the Hilbert modular form over $GL_2(\mathbb{A}_F)$. Note that any character $\psi : T(\widehat{\mathbb{Z}}) \longrightarrow \mathbb{C}^{\times}$ which is continuous is of finite order, and we have an ideal $\mathfrak{c}(\psi)$ which is maximal among the integral ideals \mathfrak{c} satisfying $\psi(x) = 1$ for all $x \in T(\widehat{\mathbb{Z}}) = \widehat{\mathcal{O}}^{\times}$ with $x - 1 \in \mathfrak{c}\widehat{\mathcal{O}}$. The ideal $\mathfrak{c}(\psi)$ is called the conductor of ψ .

The character $\varepsilon_+ : Z(\mathbb{A})/Z(\mathbb{Q}) \longrightarrow \mathbb{C}^{\times}$ is an arithmetic Hecke character such that $\varepsilon_+(z) = \varepsilon_1(z)\varepsilon_2(z)$ for $z \in Z(\widehat{\mathbb{Z}})$ and $\varepsilon_+(x_{\infty}) = x^{-(\kappa_1 + \kappa_2) + I}$. The infinity type of ε_+ is therefore $I - \kappa_1 - \kappa_2$. Then the conductor $\mathfrak{c}(\varepsilon_+)$ is defined in the same manner as above by taking the restriction to $Z(\widehat{\mathbb{Z}}) \cong T(\widehat{\mathbb{Z}})$. Then we define $\mathfrak{c}(\varepsilon) = \mathfrak{c}(\varepsilon_1)\mathfrak{c}(\varepsilon_2) \subset \mathfrak{c}(\varepsilon_+)$. Note that the two characters $\varepsilon_1, \varepsilon_2$ are purely local and may not extend to the Hecke characters of the idele class group $F_{\mathbb{A}}^{\times}/F^{\times}$.

Now put $\varepsilon^{-} = \varepsilon_1\varepsilon_2^{-1}$, and assume that ε^{-} factors through $(\mathcal{O}/\mathfrak{N})^{\times}$, i.e., $\mathfrak{c}(\varepsilon^{-}) \supset \mathfrak{N}$. Consider

the standard level group

$$(3) \quad \widehat{\Gamma}_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\widehat{\mathbb{Z}}) \mid c \equiv 0 \pmod{\mathfrak{N}\widehat{\mathcal{O}}} \right\}.$$

$$(4) \quad \widehat{\Gamma}(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widehat{\Gamma}_0(\mathfrak{N}) \mid a, d \equiv 1 \pmod{\mathfrak{N}\widehat{\mathcal{O}}} \right\}.$$

The characters ε^- and ε_2 induce a continuous character of the compact group $\widehat{\Gamma}_0(\mathfrak{N})$ which we also denote by ε and defined by

$$(5) \quad \varepsilon : \widehat{\Gamma}_0(\mathfrak{N}) \longrightarrow \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \varepsilon_2(ad - bc)\varepsilon^-(a_{\mathfrak{N}}) = \varepsilon_1(ad - bc)(\varepsilon^-)^{-1}(d_{\mathfrak{N}}).$$

Now assume that $\kappa_1 + \kappa_2 = [\kappa]I$, and define a factor of automorphy associated to κ as follows:

$$(6) \quad J_\kappa(g, z) = \det(g)^{\kappa_1 - I} j(g, z)^{\kappa_2 - \kappa_1 + I}, \text{ for } g \in G(\mathbb{R}) \text{ and } z \in \mathfrak{H}^I,$$

where, for $g = (g_\sigma) \in GL_2(\mathbb{R})^I = GL_2(F_\infty)$ and $z = (z_\sigma) \in \mathfrak{H}^I$; $j(g, z) = (c_\sigma z_\sigma + d_\sigma)_{\sigma \in I} \in \mathbb{C}^I = F \otimes_{\mathbb{R}} \mathbb{C}$. Then $S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C})$ is defined to be the space of functions $f : G(\mathbb{A}) \longrightarrow \mathbb{C}$ satisfying the following three conditions.

(S1) $f(\alpha xuz) = \varepsilon_+(z)\varepsilon(u)f(x)J_\kappa(u_\infty, \mathbf{i})^{-1}$, for all $\alpha \in G(\mathbb{Q})$, $z \in Z(\mathbb{A})$, and $u \in \Gamma_0(\mathfrak{N})C_{\mathbf{i}}$, for the stabilizer $C_{\mathbf{i}}$ of $\mathbf{i} = (\sqrt{-1}, \dots, \sqrt{-1}) \in \mathfrak{H}^I$ in $G(\mathbb{R})^+ =$ identity connected component of $G(\mathbb{R})$.

(S2) for any $u \in G(\mathbb{R})$ with $u(\mathbf{i}) = z$ for every $z \in \mathfrak{H}^I$, the function $f_g : \mathfrak{H}^I \longrightarrow \mathbb{C}$ defined by $f_g(z) = f(gu_\infty)J_\kappa(u_\infty, \mathbf{i})$ for each $g \in G(\mathbb{A}^{(\infty)})$, is a holomorphic function on \mathfrak{H}^I for every g ;

(S3) for every z , the function $f_g(z)$ is rapidly decreasing as $\text{Im}(z_\sigma) \longrightarrow \infty$ for all $\sigma \in I$ uniformly.

A function in $S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C})$ is called a *Hilbert cusp form* of level \mathfrak{N} and character ε . It is easy to check that the function f_g satisfies the classical automorphy condition ([Hid06, 2.3.5]):

$$(7) \quad f_g(\gamma(z)) = \varepsilon^{-1}(g^{-1}\gamma g)f_g(z)J_\kappa(\gamma, z), \text{ for all } \gamma \in \Gamma_g(\mathfrak{N}),$$

where $\Gamma_g(\mathfrak{N}) = g\Gamma_0(\mathfrak{N})g^{-1}G(\mathbb{R})^+ \cap G(\mathbb{Q})$.

Now consider the level \mathfrak{N} semigroup of level $\Delta_0(\mathfrak{N}) \subset M_2(\widehat{\mathcal{O}}) \cap G(\mathbb{A}^{(\infty)})$ by

$$(8) \quad \Delta_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\widehat{\mathcal{O}}) \cap G(\mathbb{A}^{(\infty)}) \mid a_{\mathfrak{N}} \in \mathcal{O}_{\mathfrak{N}}^\times, c \in \mathfrak{N}\widehat{\mathcal{O}} \right\},$$

where $\mathcal{O}_{\mathfrak{N}} = \prod_{\mathfrak{l} \mid \mathfrak{N}} \mathcal{O}_{\mathfrak{l}}$ with \mathfrak{l} running over primes dividing \mathfrak{N} . The opposite semigroup $\Delta_0^*(\mathfrak{N})$ is defined to be

$$(9) \quad \Delta_0^*(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\widehat{\mathcal{O}}) \cap G(\mathbb{A}^{(\infty)}) \mid d_{\mathfrak{N}} \in \mathcal{O}_{\mathfrak{N}}^\times, c \in \mathfrak{N}\widehat{\mathcal{O}} \right\}.$$

We now extend the character ε_2 to $T(\mathbb{A}^{(\infty)})$ by trivially extending on $\oplus \mathfrak{q}\varpi^{\mathbb{Z}}$ and then extend ε_1 to $T(\mathbb{A}^{(\infty)})$ by $\varepsilon_1\varepsilon_2(x) = \varepsilon_+(x^{(\infty)})$. We put $\varepsilon^-(a) = \varepsilon_2^{-1}(a)\varepsilon_1(a)$ for all $a \in T(\mathbb{A}^{(\infty)})$.

Next extend the character ε of $\widehat{\Gamma}_0(\mathfrak{N})$ in (5) to the semigroup $\Delta_0(\mathfrak{N})$ by

$$(10) \quad \varepsilon \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \varepsilon_1(ad - bc)(\varepsilon^-)^{-1}(d_{\mathfrak{N}}).$$

Let $f \in S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C})$ be a Hilbert modular form. The Hecke operator $T(y)$ of the double coset $\widehat{\Gamma}_0(\mathfrak{N}) \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \widehat{\Gamma}_0(\mathfrak{N}) = \sqcup_{\delta} \widehat{\Gamma}_0(\mathfrak{N})$ is defined by

$$(11) \quad f \mid T(y)(g) = \sum_{\delta} \varepsilon(\delta)^{-1} f(g\delta).$$

This operator preserves the space $S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C})$. Then, as in [Hid06, 4.3], the operator $\mathbb{T}(y) = y_p^{-\kappa_1} T(y)$ is optimally p -integral. If f is a Hecke eigenform, then the eigenvalue $a(y, f)$ of $T(y)$ depends only on the ideal $\mathfrak{n} = y\widehat{\mathcal{O}} \cap F$. Therefore, for each prime \mathfrak{l} of F , we write $a(\mathfrak{l}, f) = a(\varpi_{\mathfrak{l}}, f)$ and we put $T(\mathfrak{l}) := T(\varpi_{\mathfrak{l}})$. Therefore the y -th Fourier coefficient of f is $\varepsilon_1(y)a(y, f)$ for each Hecke eigenform f normalized so that $c(1, f) = 1$, and the Fourier coefficient depends on y (if $\varepsilon_1 \neq 1$) and not just on the ideal \mathfrak{n} .

A $T(\mathfrak{p})$ -eigenform f has \mathfrak{p} -slope equal to 0 if the absolute value $|y_p^{-\kappa_1} a(\mathfrak{p}, f)|_p = 1$. A \mathfrak{p} -slope 0-form can have positive slope at primes $\mathfrak{p}' \mid p$ different from \mathfrak{p} . For a Hecke eigenform $f \in S_\kappa(\mathfrak{N}\mathfrak{p}^{r+1}, \varepsilon; \mathbb{C})$ ($\mathfrak{p} \nmid \mathfrak{N}, r \geq 0$) and a subfield K of $\overline{\mathbb{Q}}$, the Hecke field $K(f)$ inside \mathbb{C} is generated over K by the eigenvalues $a(\mathfrak{l}, f)$ for the Hecke operators $T(\mathfrak{l})$ for all primes \mathfrak{l} and the values of ε over finite fields.

We now recall the Fourier expansion of adelic modular forms (cf. [Hid06, Prop 2.26]). Recall the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Also recall the differential idele $d \in F_{\mathbb{A}}^\times$ with $d^{(\mathfrak{o})} = 1$ and $d\widehat{\mathcal{O}} = \mathfrak{o}\widehat{\mathcal{O}}$. Then every $f \in S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C})$ has a Fourier expansion:

$$(12) \quad f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbb{A}} \sum_{0 < \xi \in F} c(\xi y d, f) (\xi y_{\infty})^{-\kappa_1} \mathbf{e}_F(i\xi y_{\infty}) \mathbf{e}_F(\xi x),$$

where $\mathbf{e}_F : F_{\mathbb{A}}/F \rightarrow \mathbb{C}^\times$ is the additive character with $\mathbf{e}_F(x_{\infty}) = \exp(2\pi i \sum_{\sigma \in I} x_{\sigma})$ for $x_{\infty} = (x_{\sigma})_{\sigma} \in \mathbb{R}^I = F \otimes_{\mathbb{Q}} \mathbb{R}$.

Let $F[\kappa]$ denote the field fixed by $\{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F) \mid \kappa\sigma = \kappa\}$, over which the character κ is rational. Let $\mathcal{O}[\kappa]$ denote the ring of integers of $F[\kappa]$. Further, let $F[\kappa, \varepsilon]$ be the field generated by the values of ε over $F[\kappa]$. For any $F[\kappa, \varepsilon]$ -algebra A inside \mathbb{C} , we define

$$(13) \quad S_\kappa(\mathfrak{N}, \varepsilon; A) = \{f \in S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C}) \mid c(y, f) \in A \text{ if } y \text{ is integral}\}.$$

There is an interpretation of $S_\kappa(\mathfrak{N}, \varepsilon; A)$ as the space of A -rational global sections of a line bundle on a variety defined over A . Therefore, by the flat base-change theorem (cf. [Hid12, Lemma 1.10.2]), we have

$$(14) \quad S_\kappa(\mathfrak{N}, \varepsilon; A) \otimes_A \mathbb{C} = S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C}).$$

Therefore, for any $\overline{\mathbb{Q}}_p$ -algebra A , we may define

$$(15) \quad S_\kappa(\mathfrak{N}, \varepsilon; A) = S_\kappa(\mathfrak{N}, \varepsilon; \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}, i_p} A.$$

By linearity, $(y, f) \mapsto c(y, f)$ extends to a function on $F_{\mathbb{A}}^\times \times S_\kappa(\mathfrak{N}, \varepsilon; A)$ with values in A . If $u \in \widehat{\mathcal{O}}^\times$, then by [Hid06, 2.3.20], we have

$$(16) \quad c(yu, f) = \varepsilon_1(u)c(y, f).$$

The formal q -expansion of an A -rational form f has values in the space of functions on $(F_{\mathbb{A}}^{(\infty)})^\times$ with values in the formal monoid algebra $A[[q^\xi]]_{\xi \in F}$ of the multiplicative semi-group F_+ made up of totally positive elements, which is defined by

$$(17) \quad f(y) = \mathcal{N}(y)^{-1} \sum_{\xi > 0} c_p(\xi y d, f) q^\xi,$$

where $\mathcal{N} : F_{\mathbb{A}}^\times / F^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ is the character given by $\mathcal{N}(y) = y_p^{-I} |y^{(\infty)}|_{\mathbb{A}}^{-1}$, and $c_p(y, f) = y_p^{-\kappa_1} c(y, f)$.

Let $\mathcal{O}[\kappa, \varepsilon]$ be the ring of integers of the field $F[\kappa, \varepsilon]$. Then for any p -adically complete $\mathcal{O}[\kappa, \varepsilon]$ -algebra A in \mathbb{C}_p , we define

$$(18) \quad S_\kappa(\mathfrak{N}, \varepsilon; A) = \{f \in S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C}_p) \mid c_p(y, f) \in A \text{ if } y \text{ is integral}\}.$$

On this space $S_\kappa(\mathfrak{N}, \varepsilon; A)$, there are Hecke operators acting (cf. [Hid06, 2.3.4]). The Hecke operators form an A -subalgebra of $\text{End}_A(S_\kappa(\mathfrak{N}, \varepsilon; A))$ generated by $T_p(y)$ for all y of the form $\prod_{\mathfrak{q}} \varpi_{\mathfrak{q}}^{e(\mathfrak{q})}$. We denote the A -subalgebra of Hecke operators by $h_\kappa(\mathfrak{N}, \varepsilon; A)$.

2.2 Hecke algebras

Consider the subgroups $U_\alpha = \widehat{\Gamma}_0(\mathfrak{N}) \cap \widehat{\Gamma}(\mathfrak{p}^\alpha)$. Then for all $\alpha \geq \beta$, we have

$$(19) \quad S_\kappa(\mathfrak{N}\mathfrak{p}^\beta, \varepsilon; A) \hookrightarrow S_\kappa(\mathfrak{N}, \varepsilon; A) \hookrightarrow S_\kappa(U_\alpha, \varepsilon; A).$$

Now let $\mathbf{\Gamma}$ denote the torsion free part of $\mathcal{O}_\mathfrak{p}^\times$ and Δ be the torsion part. Then $\mathcal{O}_\mathfrak{p}^\times = \mathbf{\Gamma} \times \Delta$ and hence $\mathbf{G} = \mathbf{\Gamma} \times \Delta \times (\mathcal{O}/\mathfrak{N}')^\times$. We then fix κ and the initial $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_+)$. We then assume that $\varepsilon_1, \varepsilon_2$ factors through $\mathbf{G}/\mathbf{\Gamma}$ factors through $\mathbf{G}/\mathbf{\Gamma} \cong \Delta \times (\mathcal{O}/\mathfrak{N}')^\times$, for some \mathfrak{p}^{r_0+1} for some prime. It is easy to see that there exists a projective system of Hecke algebras $\{h_\kappa(U, \varepsilon; A)\}_U$, where U runs over all the open subgroups $\widehat{\Gamma}_0(\mathfrak{N}\mathfrak{p}^{r+1})$. When $\kappa_2 - \kappa_1 \geq I$, we get the universal Hecke algebra $\mathbf{h}_\kappa(\mathfrak{N}\mathfrak{p}^\infty, \varepsilon; A) = \varprojlim_U h_\kappa(U, \varepsilon; A)$.

Note that the character defined by $T : \widehat{\mathcal{O}}^\times \rightarrow \mathbf{h}_\kappa(\mathfrak{N}^\infty, \varepsilon; A)$ which maps an element u to the Hecke operator $T(u)$, factors through $\mathbf{\Gamma} = \mathbf{G}/(\Delta \times (\mathcal{O}/\mathfrak{N}')^\times)$ and induces a canonical algebra structure of $\mathbf{h}_\kappa(\mathfrak{N}\mathfrak{p}^\infty, \varepsilon; A)$ over $A[[\mathbf{\Gamma}]]$.

Suppose that W is a sufficiently large complete discrete valuation ring inside $\overline{\mathbb{Q}}_p$ containing the values of ε . We set $\mathbf{\Lambda} = W[[\mathbf{\Gamma}]]$, and let $W[\varepsilon] \subset \overline{\mathbb{Q}}_p$ be the W -subalgebra generated by the values of ε (over the finite adeles). For the Hecke operator $\mathbb{T}(\varpi_\mathfrak{p})$, one considers the nearly \mathfrak{p} -ordinary projector $e = \lim_n \mathbb{T}(\varpi_\mathfrak{p})^{n!}$. The limit is independent of the choice of $\varpi_\mathfrak{p}$. Now consider the *nearly-ordinary* Hecke algebras $h_\kappa^{\text{n.ord}}(\mathfrak{N}\mathfrak{p}^r, \varepsilon; W) := e(h_\kappa(\mathfrak{N}\mathfrak{p}^r, \varepsilon; W))$ and $\mathbf{h}_\kappa^{\text{n.ord}} = \mathbf{h}_\kappa^{\text{n.ord}}(\mathfrak{N}\mathfrak{p}^\infty, \varepsilon; W) = \varprojlim_r h_\kappa^{\text{n.ord}}(\mathfrak{N}\mathfrak{p}^r, \varepsilon; W)$. If the weight $\kappa = (0, I)$, then we set (cf. [Hid06, 3.1.5]):

$$h^{\text{n.ord}}(\mathfrak{N}, \varepsilon; W) := h_{(0, I)}^{\text{n.ord}}(\mathfrak{N}, \varepsilon; W).$$

Definition 2.1. Recall, from section 2.1, that $\kappa = (\kappa_1, \kappa_2)$ induces a character on the torus T_G . If $\kappa_2 - \kappa_1 + I \geq I$, then the pair (κ, ε) is called *arithmetic*. For an integral domain \mathbb{I} finite and flat over $W[[T_G(\mathbb{Z}_p)]]$, if a W -algebra homomorphism $(P : \mathbb{I} \rightarrow W) \in \text{Spf}(\mathbb{I})(W)$ coincides with an arithmetic weight on an open subgroup of $T_G(\mathbb{Z}_p)$, then P is referred to as an *arithmetic* point. The set of arithmetic points of $\text{Spf}(\mathbb{I})$ with values in W is denoted by $\text{Spf}^{\text{arith}}(\mathbb{I})(W)$.

Let Σ_p denote the set of primes of F lying above p . A pair (κ, ε) , with $\kappa \in X(T_G)$ such that κ_j, ε_j factors through the local norm maps

$$(20) \quad \begin{aligned} T(\mathbb{Z}_p) &\longrightarrow \prod_{\mathfrak{p}|p} \mathbb{Z}_p^\times \\ N_p((x_\mathfrak{p})_\mathfrak{p}) &= (N_\mathfrak{p}(x_\mathfrak{p}))_\mathfrak{p}, \text{ where } N_\mathfrak{p}(x_\mathfrak{p}) = N_{F_\mathfrak{p}/\mathbb{Q}_p}(x_\mathfrak{p}). \end{aligned}$$

is called *locally cyclotomic*. If (κ, ε) factor through the global norm map $N_{F/\mathbb{Q}} : T(\mathbb{Z}_p) \rightarrow \mathbb{G}_m(\mathbb{Z}_p) = \mathbb{Z}_p^\times$, then (κ, ε) is called *cyclotomic*.

The pair (κ, ε) induces a character $T_G(\widehat{\mathbb{Z}}) \rightarrow W^\times$ given by

$$(21) \quad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \varepsilon_1(a) a_p^{-\kappa_1} \varepsilon_2(d) d_p^{-\kappa_2}.$$

This further induces a W -algebra homomorphism $\pi_{\kappa, \varepsilon} : W[[T_G(\mathbb{Z}_p)]] \rightarrow W$ induced by the restriction of this character to $T_G(\mathbb{Z}_p)$. A W -point P of the formal spectrum $\text{Spf}(W[[T_G(\mathbb{Z}_p)]]))$ is called *arithmetic* if $P = \ker(\pi_{\kappa, \varepsilon})$ with $\kappa_2 - \kappa_1 - I \geq 0$. Similarly, an arithmetic point $P \in \text{Spec}(W[[T_G(\mathbb{Z}_p)]])(W)$ associated with (κ, ε) is called *locally cyclotomic* (resp. *cyclotomic*) if (κ, ε) is locally cyclotomic (resp. cyclotomic). Thus locally cyclotomic (resp. cyclotomic) points are arithmetic. Let \mathbb{I} be an integral domain which is an algebra over $W[[T_G(\mathbb{Z}_p)]]$. Then a point $P \in \text{Spec}(\mathbb{I})(W)$ is said to be *locally cyclotomic* (resp. *cyclotomic*) if the structure homomorphism $W[[T_G(\mathbb{Z}_p)]] \rightarrow \mathbb{I}$ is locally cyclotomic (resp. cyclotomic).

2.3 Locally cyclotomic Hecke algebra

Let $\Gamma_{\mathfrak{p}}$ be the p -Sylow subgroup of $\text{Gal}(F_{\mathfrak{p}}^{unr}(\mu_{p^\infty})/F_{\mathfrak{p}}^{unr})$. Then the cyclotomic character

$$\mathcal{N} : \text{Gal}(F_{\mathfrak{p}}^{unr}(\mu_{p^\infty})/F_{\mathfrak{p}}^{unr}) \longrightarrow \mathbb{Z}_p^\times$$

induces an embedding $\Gamma_{\mathfrak{p}} \hookrightarrow 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. Set $\Gamma_F = \prod_{\mathfrak{p}|p} \Gamma_{\mathfrak{p}}$.

Consider the following isomorphism

$$\begin{aligned} T(\mathbb{Z}/p^r\mathbb{Z})^2 &\rightarrow \widehat{\Gamma}_0(p^r)/\widehat{\Gamma}_1(p^r) \\ (a, d) &\mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}. \end{aligned}$$

Since $T(\mathbb{Z}/p^r\mathbb{Z}) \cong (\mathcal{O}/p^r\mathcal{O})^\times \cong \prod_{\mathfrak{p}} (\mathcal{O}_{\mathfrak{p}}/p^r\mathcal{O}_{\mathfrak{p}})^\times$; so the local norm map $N_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}}^\times \longrightarrow \mathbb{Z}_p^\times$ induces the map $N_p = \prod_{\mathfrak{p}|p} N_{\mathfrak{p}} : T(\mathbb{Z}/p^r\mathbb{Z}) \longrightarrow \prod_{\mathfrak{p}|p} (\mathbb{Z}/p^r\mathbb{Z})^\times$ for each $r > 0$. Let $S_{cyc}(p^r)$ be a subgroup of $G(\mathbb{A}^{(\infty)})$ with $\widehat{\Gamma}_0(p^r) \supset S_{cyc}(p^r) \supset S_{cyc}(p^r) \supset \widehat{\Gamma}(p^r)$ which is given by

$$(22) \quad S_{cyc}/\widehat{\Gamma}(p^r) = \ker \left(N_p^2 : T(\mathbb{Z}/p^r\mathbb{Z})^2 \longrightarrow \prod_{\mathfrak{p}|p} ((\mathbb{Z}/p^r\mathbb{Z})^\times)^2 \right).$$

Putting $S_r = S_{cyc}(p^r) \cap \widehat{\Gamma}_0(\mathfrak{N})$. Let $S(S_n, \varepsilon; A)$ denote the space of cusp forms of weight $(0, I)$ defined over the congruence subgroup S_n . A more general definition is given in [Hid06, page 165]. For $m > n$, there is an inclusion $S(S_n, \varepsilon; A) \hookrightarrow S(S_m, \varepsilon; A)$, which is compatible with the Hecke operators. By duality, this induces a W -algebra homomorphism $h(S_m, \varepsilon; W) \longrightarrow h(S_n, \varepsilon; W)$. The *universal locally cyclotomic Hecke algebra* is defined to be

$$(23) \quad \mathbf{h}_{cyc}^{n, \text{ord}}(\mathfrak{N}, \varepsilon; W[[\Gamma_F]]) := \varprojlim_n h^{n, \text{ord}}(S_n, \varepsilon; W).$$

For the character $\underline{\varepsilon} : Z(\widehat{A}^{(\infty)})\widehat{\Gamma}_0(\mathfrak{N}) \longrightarrow A^\times$, defined by $\underline{\varepsilon}(u) = \varepsilon_2(\det(u))\varepsilon^-(a_{\mathfrak{N}})\varepsilon_+(z)$ for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widehat{\Gamma}_0(\mathfrak{N})$, $z \in Z(\mathbb{A}^{(\infty)})$, assume the following condition ([Hid06, page 165]):

(sm0) $\underline{\varepsilon}$ restricted to $\Gamma_j/Z(\mathbb{Q})$ has order prime to p .

Under this condition, $\mathbf{h}_{cyc}^{n, \text{ord}}(\mathfrak{N}, \varepsilon; W[[\Gamma_F]])$ is a torsion-free $W[[\Gamma_F]]$ -module of finite type.

2.4 Modular Galois representations

Recall the following Hecke operators on $S_\kappa(\mathfrak{N}, \varepsilon; W)$:

$$(24) \quad T(y) = y_p^{-\kappa_1} \left[\widehat{\Gamma}_0(\mathfrak{N}) \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \widehat{\Gamma}_0(\mathfrak{N}) \right], \text{ if the ideal } y\mathcal{O} \text{ is prime to } \mathfrak{N}$$

$$(25) \quad U(y) = y_p^{-\kappa_1} \left[\widehat{\Gamma}_0(\mathfrak{N}) \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \widehat{\Gamma}_0(\mathfrak{N}) \right], \text{ if } y \in \widehat{\mathcal{O}}_{\mathfrak{N}}.$$

Let $f \in S_\kappa(\mathfrak{N}, \varepsilon; W)$, with $\kappa = (0, I)$ be a Hecke eigenform, and $\lambda : h_\kappa(\mathfrak{N}, \varepsilon; W) \longrightarrow \overline{\mathbb{Q}}_p$ be an algebra homomorphism satisfying $f \mid T(\varpi_{\mathfrak{q}}) = \lambda(T(\varpi_{\mathfrak{q}}))f$ for all prime ideals \mathfrak{q} . Suppose $P = \ker(\lambda)$. Recall the character $\varepsilon^- = \varepsilon_1\varepsilon_2^{-1}$. We consider the following condition:

(sf) $\mathfrak{N}/\mathfrak{c}(\varepsilon^-)$ is square-free and is prime to $\mathfrak{c}(\varepsilon^-)$.

The duality between the space of cusp forms and Hecke algebras gives rise to an algebra homomorphism $\pi_f \in \text{Hom}_{\text{alg}}(h_\kappa(\mathfrak{N}, \varepsilon; W), W)$ ([Hid06, Theorem 2.28]). We have the following theorem due to Shimura, Deligne, Serre, Carayol, Ohta, Wiles, Blasius, Rogawski, Taylor, and the version that we give here is from [Hid06, Theorem 2.43].

Theorem 2.2. Let $h = h_\kappa(\mathfrak{N}, \varepsilon; W)$ and $k(P)$ denote the field of fractions of h/P . Then there exists a continuous semi-simple Galois representation $\rho_f : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow GL_2(k(P))$, such that

- (i) ρ_f is unramified outside $p\mathfrak{N}$,
- (ii) $\text{Tr}(\rho_f(\text{Frob}_l)) = \lambda(T(\varpi_l))$ for $l \nmid p\mathfrak{N}$.

Let f be nearly ordinary at all primes $\mathfrak{p} \mid p$. Then we have the following theorem ([Hid06, Theorem 2.43 (3)]).

Theorem 2.3. Let f be nearly ordinary at all primes $\mathfrak{p} \mid p$, i.e., $f|_{U_p(\varpi_{\mathfrak{p}})} = \lambda(U_p(\varpi_{\mathfrak{p}}))f$ with a p -adic unit $\lambda(U_p(\varpi_{\mathfrak{p}}))$. Then

$$(26) \quad \rho_f|_{D_{\mathfrak{p}}} \cong \begin{pmatrix} \epsilon_{\mathfrak{p}} & * \\ 0 & \delta_{\mathfrak{p}} \end{pmatrix}$$

for the decomposition subgroup $D_{\mathfrak{p}}$ at \mathfrak{p} , and $\delta_{\mathfrak{p}}([\varpi_{\mathfrak{p}}, F_{\mathfrak{p}}]) = \lambda(U_p(\varpi_{\mathfrak{p}}))$, for the local Artin symbol $[\varpi_{\mathfrak{p}}, F_{\mathfrak{p}}]$. In particular, $\delta_{\mathfrak{p}}([\varpi_{\mathfrak{p}}, F_{\mathfrak{p}}]) = \varepsilon_{1,\mathfrak{p}}(u)u^{-\kappa_1}$ for $u \in \mathcal{O}_{\mathfrak{p}}^\times$.

2.5 Locally cyclotomic deformation

We now recall the locally cyclotomic deformation associated to a Hecke eigenform from [Hid06, 3.2.1, 3.2.2]. Let $f_0 \in S(\mathfrak{N}, \varepsilon; B)$ be a Hecke eigenform, whose Galois representation $\rho_0 = \rho_{f_0} : \text{Gal}_F \rightarrow GL_2(W)$ is unramified outside $p\mathfrak{c}(\varepsilon)$, where W as before is the p -adic completion of B . Among all the forms equivalent to f_0 , we assume that f_0 has the maximal level. Recall the following condition

(sf) $\mathfrak{N}_0 = \mathfrak{N}/\mathfrak{c}(\varepsilon^-)$ is square-free and is prime to $\mathfrak{c}(\varepsilon^-)$.

Let f_0 be nearly ordinary at all prime ideals $\mathfrak{p} \mid p$. Recall the following conditions on the representation $\bar{\rho} := (\rho_0 \bmod \mathfrak{m}_B) : \text{Gal}_F \rightarrow GL_2(F)$. These are the conditions (h1) – (h4) in [Hid06, page 185].

(h1) ε has order prime to p .

(h2) $\rho_0|_{D_{\mathfrak{p}}}$ is reducible for all $\mathfrak{p} \mid p$.

(h3) For all $\mathfrak{p} \mid p$ in F , viewing the local representations at \mathfrak{p} is given by $\pi_{0,\mathfrak{p}} \cong \pi(\eta_{1,\mathfrak{p}}, \eta_{2,\mathfrak{p}})$ or $\pi_{0,\mathfrak{p}} \cong \sigma(\eta_{1,\mathfrak{p}}, \eta_{2,\mathfrak{p}})$ with $\eta_{2,\mathfrak{p}} = \eta_{1,\mathfrak{p}} \cdot |\cdot|_{\mathfrak{p}}^{-1}$. Then by local class field theory, we have

$$\rho_0|_{D_{\mathfrak{p}}} \cong \begin{pmatrix} \eta_{1,\mathfrak{p}}^{-1} \varepsilon_{+, \mathfrak{p}} \mathcal{N}_{\mathfrak{p}} & * \\ 0 & \eta_{1,\mathfrak{p}} \end{pmatrix}$$

with $\bar{\delta}_{\mathfrak{p}}^{-1} \det(\bar{\rho}) \neq \bar{\delta}_{\mathfrak{p}}$, and $\bar{\delta}_{\mathfrak{p}} = (\eta_{1,\mathfrak{p}} \bmod \mathfrak{m}_B)$. The finite order Hecke character ε_+ is regarded as a global Galois character by class field theory, and $\mathcal{N}_{\mathfrak{p}}$ is the p -adic cyclotomic character restricted to $D_{\mathfrak{p}}$. The character $\bar{\delta}_{\mathfrak{p}}$ (resp. $\eta_{1,\mathfrak{p}}$) is the *nearly \mathfrak{p} -ordinary character* of $\bar{\rho}$ (resp. ρ_0).

(h4) If \mathfrak{q} is a prime ideal such that $\mathfrak{q} \nmid p$ but $\mathfrak{q} \mid \mathfrak{N}/\mathfrak{c}(\varepsilon^-)$, then $\bar{\rho}|_{D_{\mathfrak{q}}}$ has ramification index divisible by p . So, $\bar{\rho}$ restricted to the inertia subgroup $I_{\mathfrak{q}}$ has the isomorphism $\bar{\rho}|_{I_{\mathfrak{q}}} \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, which is non-semisimple with a non-trivial $*$.

Now we consider the following deformation functor on the category CNL_W of complete, noetherian local W -algebras A such that $A/\mathfrak{m}_A \cong \mathbb{F}$. The deformation functor that we are considering is the one that is denoted Φ^{cyc} in [Hid06, 3.2.8]. We recall that the functor $\Phi^{cyc} : \text{CNL}_W \rightarrow \text{SETS}$ is defined to be the set of isomorphism classes of representations $\rho : \text{Gal}_F \rightarrow GL_2(A)$ satisfying the following conditions:

(L1) $\rho \bmod \mathfrak{m}_A \cong \bar{\rho}$.

(L2) ρ is unramified outside $\mathfrak{c}(\varepsilon)\mathfrak{N}$.

(L3) $\det(\rho) = \varepsilon_+ \mathcal{N}$, for the global cyclotomic character \mathcal{N} .

(L4) ρ is nearly ordinary at all $\mathfrak{p} \mid p$, and $\rho|_{D_{\mathfrak{p}}} \cong \begin{pmatrix} \epsilon_{\mathfrak{p}} & * \\ 0 & \delta_{\mathfrak{p}} \end{pmatrix}$ such that the characters $\epsilon_{\mathfrak{p}}|_{I_{\mathfrak{p}}} \varepsilon_{2,\mathfrak{p}}^{-1}$ and $\delta_{\mathfrak{p}}|_{I_{\mathfrak{p}}} \varepsilon_{1,\mathfrak{p}}^{-1}$ factor through $\text{Gal}(F_{\mathfrak{p}}^{\text{unr}}(\mu_{p^\infty})/F_{\mathfrak{p}}^{\text{unr}})$ for all primes $\mathfrak{p} \mid p$.

(L5) $\rho|_{D_{\mathfrak{q}}} \cong \begin{pmatrix} \epsilon_{\mathfrak{q}} & * \\ 0 & \delta_{\mathfrak{q}} \end{pmatrix}$ with $\delta_{\mathfrak{q}} \bmod \mathfrak{m}_A \cong \bar{\delta}_{\mathfrak{q}}$ and $\delta_{\mathfrak{q}}|_{I_{\mathfrak{q}}} = \varepsilon_{1,\mathfrak{q}}$ if $\mathfrak{q} \mid \mathfrak{c}(\varepsilon)\mathfrak{N}$ and $\mathfrak{q} \nmid p$; and $\rho|_{I_{\mathfrak{q}}} \otimes \varepsilon_1^{-1}$ is unramified if $\mathfrak{q} \mid \mathfrak{c}(\varepsilon)$ and $\mathfrak{q} \nmid p\mathfrak{N}$.

The conditions (L1)-(L3) correspond to the conditions (Q1)-(Q3) and the last condition (L5) is the condition (Q6) in [Hid06, page 186]. The condition (L4) is the condition (Q4') in [Hid06, 3.2.8]. By Mazur's theorem, (see for example [Hid06, Theorem 1.46]), the functor Φ^{cyc} is represented by a universal couple $(\mathcal{R}_F, \rho_F^{cyc})$, with $\mathcal{R}_F \in \text{CNL}_W$.

3 Deformation rings and base change

3.1 Base change of deformation rings

Let E be a totally real field and $f_0 \in S_{\kappa}(\mathfrak{N}, \varepsilon; W)$ with $\kappa = (0, I)$ be a Hilbert modular eigenform defined over the totally real field E . Let ρ_E denote the Galois representation that is attached to f_0 . We assume that the Galois representation ρ_E satisfies the conditions (h1)-(h4) in section 2.5.

Let F be a finite totally real Galois extension of E such that the Galois group $\Delta := \text{Gal}(F/E)$ is a *finite* p group. The Galois group Δ is *not necessarily* cyclic. Since Δ is a group with order a power of p , it is a *solvable* group. Therefore the Hecke eigenform $f_0 \in S_{\kappa}(\mathfrak{N}, \varepsilon; W)$ admits a *unique* base-change lift, say, $f \in S_{\kappa}(\mathfrak{N}', \varepsilon; W)$, which is defined over the totally real field F (cf. [Hid06, 3.3.3]), for an appropriate choice of \mathfrak{N}' , such that the associated Galois representation $\rho_f : \text{Gal}_F \rightarrow GL_2(W)$ is equivalent to the restriction $\rho_E|_{\text{Gal}_F}$. If the representation ρ_E satisfies the conditions (h1)-(h4) over E , then the restriction ρ_F satisfies the conditions (h1)-(h4) over F .

Let Φ_E^{cyc} denote the locally cyclotomic deformation for $\bar{\rho}_E$, and Φ_F^{cyc} denote the locally cyclotomic deformation for $\bar{\rho}_F$. Since $\bar{\rho}_E$ is absolutely irreducible and Δ is a finite p -group, by [Hid06, Lemma 1.62], the representation $\bar{\rho}_F$ is absolutely irreducible. Therefore, Φ_F^{cyc} is representable. Let $(\mathcal{R}_F, \rho_F^{cyc})$ denote the universal deformation ring and the universal deformation of $\bar{\rho}_F$.

Then $\rho_E^{cyc}|_{\text{Gal}_F}$ is a deformation in Φ_F^{cyc} . Therefore we have a non-trivial algebra homomorphism $\alpha : \mathcal{R}_F \rightarrow \mathcal{R}_E$ such that $\alpha \circ \rho_F^{cyc} \cong \rho_E^{cyc}|_{\text{Gal}_F}$. The morphism α is referred to as the base change morphism. We now describe the action of Δ on Φ_F^{cyc} and \mathcal{R}_F .

Δ -action on Φ_F^{cyc} : Let $\sigma \in \Delta$ and $\rho \in \Phi_F^{cyc}(A)$, where A is an \mathcal{O} -algebra in $\text{CNL}_{\mathcal{O}}$. Consider any $c(\sigma) \in GL_2(\mathcal{O})$ such that $c(\sigma) \equiv \bar{\rho}(\sigma) \pmod{\mathfrak{m}_{\mathcal{O}}}$. Then the action of σ on ρ is defined by

$$(27) \quad \rho^{\sigma}(g) := c(\sigma)^{-1} \rho(\sigma g \sigma^{-1}) c(\sigma) \in \Phi_F^{cyc}(A).$$

The strict equivalence class of ρ^{σ} is well defined and depends only upon the class of σ in Δ . This gives a well-defined action of Δ on Φ_F^{cyc} .

Δ -action on \mathcal{R}_F : For any $\sigma \in \Delta$, since $\overline{\rho^{\sigma}} = \overline{\rho}$, \mathcal{R}_F is the universal deformation ring for ρ^{σ} . Therefore, there is a morphism $\mathcal{R}_F \xrightarrow{\tilde{\sigma}} \mathcal{R}_F$ in $\text{CNL}_{\mathcal{O}}$. Similarly, we have $\mathcal{R}_F \xrightarrow{(\tilde{\sigma}^{-1})} \mathcal{R}_F$ in $\text{CNL}_{\mathcal{O}}$. Composing these two morphisms gives the identity, so that $\tilde{\sigma}$ is an automorphism in $\text{CNL}_{\mathcal{O}}$. Extending this to $\mathcal{O}[\Delta]$, we find that \mathcal{R}_F is a module over $\mathcal{O}[\Delta]$. In other words, the universality of the deformation rings gives rise to the automorphisms which define the action.

3.2 Control of deformation rings

Recall that Δ is a group of order a power of p . Consider the base-change morphism $\mathcal{R}_F \xrightarrow{\alpha} \mathcal{R}_E$ in $CNL_{\mathcal{O}}$. We consider the following ideal

$$(28) \quad I_{\Delta}(\mathcal{R}_F) := \langle \sigma x - x \mid x \in \mathcal{R}_F, \sigma \in \Delta \rangle$$

which is the augmentation ideal. This is an ideal of \mathcal{R}_F . Let $(\mathcal{R}_F)_{\Delta} := \mathcal{R}_F / I_{\Delta}(\mathcal{R}_F)$. Since the determinant of the locally cyclotomic deformation functor is fixed, by [Hid00b, Prop 5.41], we have the following proposition

Proposition 3.1. $(\mathcal{R}_F)_{\Delta} \cong \mathcal{R}_E$.

4 Adjoint Selmer groups and Kähler differentials

4.1 Selmer groups

Let p be an odd prime. We fix an algebraic closure $\bar{\mathbb{Q}}$ and embeddings $\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}$ and $\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}_l$ for every prime l . For a prime p in \mathbb{Q} , let D_p denote the decomposition group under this embedding. For a prime \mathfrak{p} in a finite extension F of \mathbb{Q} , let $D_{\mathfrak{p}}$ denote the decomposition group at \mathfrak{p} defined by the above embedding. Let \mathcal{O} be a finite extension of \mathbb{Z}_p and $CNL_{\mathcal{O}}$ denote the category of *complete noetherian local rings which are \mathcal{O} -algebras*.

We recall the definition of the *adjoint representation* associated to a 2-dimensional Galois representation. Let $\mathbb{L} \in CNL_{\mathcal{O}}$ and \mathbb{M} be the quotient field of \mathbb{L} . Consider a two-dimensional representation $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{L})$. Let $\mathbb{L} := \mathbb{L}^2$. Then ρ induces an action of $G_{\mathbb{Q}}$ on $M_2(\mathbb{L})$, the ring of 2×2 -matrices over \mathbb{L} , by conjugation, i. e., $\sigma(x) := \rho(\sigma)x\rho(\sigma)^{-1}$. Then the *adjoint representation* is defined by

$$(29) \quad \text{Ad}^0(\rho) := \{\eta \in \text{End}_{\mathbb{L}}(\mathbb{L}) \mid \text{Trace}(\eta) = 0\}.$$

It is easy to see that this is a 3-dimensional representation of $G_{\mathbb{Q}}$.

Definition 4.1. Let D_p denote the decomposition group at p . Then the representation ρ is said to be *nearly ordinary* at p , if there is a two-step filtration of \mathbb{L} given by

$$(30) \quad \mathbb{L} \supset \mathcal{F}_p^+ \mathbb{L} \supset 0$$

as D_p -modules, such that $\mathcal{F}_p^+ \mathbb{L}$ is *free* of rank *one* over \mathbb{L} .

If ρ is nearly ordinary, then it induces on $\text{Ad}^0(\rho)$ the following three-step filtration stable under D_p :

$$(31) \quad \text{Ad}^0(\rho) \supset \mathcal{F}_p^- \text{Ad}^0(\rho) \supset \mathcal{F}_p^+ \text{Ad}^0(\rho) \supset 0$$

where

$$(32) \quad \begin{aligned} \mathcal{F}_p^- \text{Ad}^0(\rho) &= \{\eta \in \text{Ad}^0(\rho) \mid \eta(\mathcal{F}_p^+ \mathbb{L}) \subset \mathcal{F}_p^+ \mathbb{L}\}, \text{ and} \\ \mathcal{F}_p^+ \text{Ad}^0(\rho) &= \{\eta \in \text{Ad}^0(\rho) \mid \eta(\mathcal{F}_p^+ \mathbb{L}) = 0\}. \end{aligned}$$

In terms of matrices, if we choose a basis of \mathbb{L} containing a generator of $\mathcal{F}_p^+ \text{Ad}^0(\rho)$ and identify $\text{End}_{\mathbb{L}}(\mathbb{L})$ with $M_2(\mathbb{L})$ using this basis, then $\mathcal{F}_p^- \text{Ad}^0(\rho)$ is made up of *upper triangular matrices with trace zero*. On the other hand, $\mathcal{F}_p^+ \text{Ad}^0(\rho)$ is made up of *upper nilpotent matrices*.

Definition 4.2. For a number field F , let $\rho_F : \text{Gal}_F \rightarrow GL_2(\mathbb{L})$ be a representation of Gal_F . Then ρ_F is *nearly ordinary at a prime \mathfrak{p} of F lying above p* if there is a two-step filtration of \mathbb{L}^2 as $D_{\mathfrak{p}}$ -modules as in (30). Then this filtration induces a filtration of $\text{Ad}^0(\rho_F)$ restricted to $D_{\mathfrak{p}}$ for each prime \mathfrak{p} , that is,

$$(33) \quad \text{Ad}^0(\rho) \supset \mathcal{F}_{\mathfrak{p}}^- \text{Ad}^0(\rho) \supset \mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho) \supset 0.$$

Recall that \mathbb{M} is the quotient field of \mathbb{L} . We put $\mathbb{V} := \mathbb{L} \otimes_{\mathbb{L}} \mathbb{M}$ and $\mathbb{A} := \mathbb{V}/\mathbb{L}$. Let $\text{Ad}^0(\mathbb{V}) := \text{Ad}^0(\rho) \otimes \mathbb{M}$ and $\text{Ad}^0(\mathbb{A}) := \text{Ad}^0(\mathbb{V})/\text{Ad}^0(\rho)$.

Let F be any algebraic extension of \mathbb{Q} . Let $\rho_F : \text{Gal}_F \rightarrow GL_2(\mathbb{L})$ be nearly ordinary at *every* prime of F above p . Then for each $\mathfrak{p} \mid p$ in F , we have the following filtration

$$(34) \quad \text{Ad}^0(\rho_F) \supset \mathcal{F}_{\mathfrak{p}}^- \text{Ad}^0(\rho_F) \supset \mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho_F) \supset \{0\}.$$

This induces the following filtration on $\text{Ad}^0(\mathbb{A})$

$$(35) \quad \text{Ad}^0(\mathbb{A}) \supset \mathcal{F}_{\mathfrak{p}}^- \text{Ad}^0(\mathbb{A}) \supset \mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\mathbb{A}) \supset \{0\}.$$

This filtration allows us to define the following *local* conditions:

$$(36) \quad \mathcal{L}(F_{\mathfrak{q}}) = \begin{cases} \ker [H^1(F_{\mathfrak{q}}, \mathbb{A}) \rightarrow H^1(F_{\mathfrak{q}}, \mathbb{A})/\mathcal{F}_{\mathfrak{q}}^+ \text{Ad}^0(\mathbb{A})] & \text{for } \mathfrak{q} \mid p, \\ \ker [H^1(F_{\mathfrak{q}}, \mathbb{A}) \rightarrow H^1(F_{\mathfrak{q}}, \mathbb{A})] & \text{for } \mathfrak{q} \nmid p. \end{cases}$$

Definition 4.3. The Selmer group of $\text{Ad}^0(\rho)$ over F is defined by

$$(37) \quad \text{Sel}_F(\text{Ad}^0(\rho)) := \ker \left[H^1(F^{\Sigma}/F, \mathbb{A}) \rightarrow \prod_{\mathfrak{q}} \frac{H^1(F_{\mathfrak{q}}, \mathbb{A})}{\mathcal{L}(F_{\mathfrak{q}})} \right].$$

Let Σ be any finite set of primes of F containing the primes above p , the infinite primes and the primes ramified in F . Let F^{Σ} denote the maximal extension of F that is unramified outside F . Suppose that F_{∞} is any pro- p , p -adic Lie extension of F that is contained in F^{Σ} and $F_{\infty, \mathfrak{q}} := \varinjlim_n F_{n, \mathfrak{q}_n}$, where F_n are finite extensions of F and $\{\mathfrak{q}_n\}$ is a compatible sequence of primes. Then by restriction we define local conditions $\mathcal{L}(F_{\infty, \mathfrak{q}})$ as in (36).

Definition 4.4. We define the *Selmer group* of $\text{Ad}^0(\rho)$ over F_{∞} by

$$(38) \quad \text{Sel}_{F_{\infty}}(\text{Ad}^0(\rho)) := \ker \left[H^1(F^{\Sigma}/F_{\infty}, \text{Ad}^0(\mathbb{A})) \rightarrow \prod_{\mathfrak{q} \mid \Sigma} \frac{H^1(F_{\infty, \mathfrak{q}}, \mathbb{A})}{\mathcal{L}(F_{\infty, \mathfrak{q}})} \right].$$

There is an action of the Galois group $\mathcal{G} := \text{Gal}(F_{\infty}/F)$ on $\text{Sel}_{F_{\infty}}(\text{Ad}^0(\rho))$ via *conjugation*: if $[c] \in H^1(F^{\Sigma}/F_{\infty}, A)$ is any cocycle class and $g \in \mathcal{G}$, then the action is given by $(g * c)(\sigma) := \tilde{g}c(\tilde{g}^{-1}\sigma\tilde{g})$, for a lift \tilde{g} of g to Gal_F .

4.2 Kähler differentials

Let A and B be complete local noetherian algebras. Let A be a B -algebra and let $\Omega_{A/B}$ denote the A -module of Kähler differentials of A over B . The Selmer group attached to the adjoint representation is related to Kähler differentials as follows (see [MT90] or [HT94]).

Consider the representation $\bar{\rho}_F$ and the locally cyclotomic deformation functor Φ_F^{cyc} .

Theorem 4.5. [Hid06, Prop 3.87] Consider the representation $\bar{\rho}_F$ that is attached to a Hilbert modular eigenform $f_0 \in S_{\kappa}(\mathfrak{N}, \varepsilon; W)$, with $\kappa = (0, I)$, and also satisfying the conditions (h1)-(h4). Let Φ_F^{cyc} be the locally cyclotomic deformation functor of $\bar{\rho}_F$. Suppose that Φ_F^{cyc} is represented by the universal deformation ring \mathcal{R}_F and ρ_F is the representation of Gal_F into $GL_2(\mathcal{R}_F)$. Then for any $A \in \text{CNL}_{\mathcal{O}}$ and $\tilde{\rho} \in \Phi_F^{cyc}(A)$, with φ denoting the morphism $\mathcal{R}_F \rightarrow A$, there exists a canonical isomorphism

$$(39) \quad \text{Sel}_F^*(\text{Ad}^0(\tilde{\rho}_F) \otimes_A A^*) \cong \Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F, \varphi} A.$$

In particular,

$$(40) \quad \text{Sel}_F^*(\text{Ad}^0(\rho_F)) \cong \Omega_{\mathcal{R}_F/W[[\Gamma_F]]}.$$

Corollary 4.7. *Let A_∞ be an \mathcal{O} -algebra with a continuous action of \mathcal{G} which is a pro-object in $CNL_{\mathcal{O}}$. Suppose that \mathcal{R}_∞ has a structure of A_∞ -algebra and that the \mathcal{G} -action on A_∞ and \mathcal{R}_∞ are compatible. Thus \mathcal{R}_C is an A_C -algebra for $A_C = (A_\infty)_C$. Let B be an algebra in $CNL_{\mathcal{O}}$ and $\pi : \mathcal{R} \rightarrow B$ be an A_∞ -algebra homomorphism. Then, for any closed subgroups $\mathcal{C} \subset \mathcal{H} \subset \mathcal{G}$, we have:*

$$(\Omega_{\mathcal{R}_C/A_C} \otimes_{\mathcal{R}_C} B)_{\mathcal{H}} \cong \Omega_{\mathcal{R}_{\mathcal{H}}/A_{\mathcal{H}}} \otimes_{\mathcal{R}_{\mathcal{H}}} B.$$

Proof. By the above proposition, we have,

$$\begin{aligned} (\Omega_{\mathcal{R}_C/A} \hat{\otimes}_{\mathcal{R}_C} B)_{\mathcal{H}} &\cong \Omega_{\mathcal{R}_{\mathcal{H}}/A} \hat{\otimes}_{\mathcal{R}_{\mathcal{H}}} B, \\ (\Omega_{A_C/\mathcal{O}} \hat{\otimes}_{A_C} B)_{\mathcal{H}} &\cong \Omega_{A_{\mathcal{H}}/\mathcal{O}} \hat{\otimes}_{A_{\mathcal{H}}} B. \end{aligned}$$

These isomorphisms give rise to the following commutative diagram:

$$\begin{array}{ccccccc} (\Omega_{A_\infty/\mathcal{O}} \hat{\otimes}_{A_\infty} B)_{\mathcal{H}} & \longrightarrow & (\Omega_{\mathcal{R}_\infty/A} \hat{\otimes}_{\mathcal{R}_\infty} B)_{\mathcal{H}} & \longrightarrow & (\Omega_{\mathcal{R}_\infty/A_\infty} \hat{\otimes}_{\mathcal{R}_\infty} B)_{\mathcal{H}} & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \Omega_{A_{\mathcal{H}}/\mathcal{O}} \hat{\otimes}_{A_{\mathcal{H}}} B & \longrightarrow & \Omega_{\mathcal{R}_{\mathcal{H}}/\mathcal{O}} \hat{\otimes}_{A_{\mathcal{H}}} B & \longrightarrow & \Omega_{\mathcal{R}_{\mathcal{H}}/A_{\mathcal{H}}} \hat{\otimes}_{\mathcal{R}_{\mathcal{H}}} B & \longrightarrow & 0. \end{array}$$

Hence

$$(\Omega_{\mathcal{R}_C/A_C} \otimes_{\mathcal{R}_C} B)_{\mathcal{H}} \cong \Omega_{\mathcal{R}_{\mathcal{H}}/A_{\mathcal{H}}} \otimes_{\mathcal{R}_{\mathcal{H}}} B.$$

□

5 Deformation rings in an *admissible* tower E_∞/E

5.1 Admissible p -adic Lie extension

In this section, we study the deformation rings of the functor Φ_F^{cyc} , when F varies over finite Galois subextensions of an *admissible p -adic Lie extension* E_∞ over E , whose definition we recall below.

Definition 5.1. An *admissible p -adic Lie extension* E_∞ of E is a Galois extension of E such that (1) E_∞/E is unramified outside a finite set of primes of E ; (2) E_∞ is totally real; (3) E_∞ is a p -adic Lie extension; and (4) $E_{cyc} \subset E_\infty$.

Let E_∞ be an *admissible p -adic Lie extension*. Let $E_\infty := \cup_n E_n$, where E_n is a finite Galois extension of E for every n . Consider the functor $\Phi_{E_n}^{cyc}$ and let \mathcal{R}_{E_n} denote the universal deformation ring over E_n . Since the extension E_n/E is a pro- p extension, the conditions (\mathbf{A}_{E_n}) and (\mathbf{Z}_{E_n}) are satisfied. Then for every n , we have the base change morphisms $\mathcal{R}_{E_n} \rightarrow \mathcal{R}_E$. Consider the projective limit

$$(45) \quad \mathcal{R}_\infty := \varprojlim_n \mathcal{R}_{E_n}.$$

The action of $\Delta_n := G(E_n/E)$ on \mathcal{R}_{E_n} induces an action of \mathcal{G} on \mathcal{R}_∞ . Moreover, for finite Galois extensions F_m inside E_{cyc} , we also define

$$(46) \quad \mathcal{R}_{cyc} := \varprojlim_m \mathcal{R}_{F_m}.$$

Proposition 5.2. *Let $\mathcal{H} := \text{Gal}(E_\infty/E_{cyc})$. Then \mathcal{H} acts on \mathcal{R}_∞ , and we have a morphism of rings $(\mathcal{R}_\infty)_{\mathcal{H}} \rightarrow \mathcal{R}_{cyc}$, which is an isomorphism of algebras.*

Proof. This follows from the the base-change isomorphism. □

Corollary 5.3. *If \mathcal{R}_∞ is noetherian ring, then \mathcal{R}_{cyc} is a noetherian ring.*

Proof. If \mathcal{R}_∞ is a noetherian ring, then $(\mathcal{R}_\infty)_\mathcal{H}$ is noetherian. \square

Remark 5.4. For the deformation rings of the nearly ordinary functor and the fixed determinant, analogous results over E_∞/E can be proven using [Hid00a, Cor 3.2].

5.2 Example over decomposition groups

Our example is a generalization of an example due to Hida in the cyclotomic case. Let E_∞ be a p -adic Lie extension which is totally ramified at all the primes of E above p and $E_\infty = \cup_n E_n$ with $E_0 = E$. Let \mathfrak{p} denote a prime of E lying above p , and we also denote by \mathfrak{p}_n the unique prime of E_n above p . Let $D_{n,\mathfrak{p}}$ be the decomposition group of \mathfrak{p} . We consider the universal nearly ordinary representation $\rho : \text{Gal}_E \rightarrow GL_2(\mathcal{R}_0)$. Then restricted to the decomposition subgroup $D_{0,\mathfrak{p}}$ at a prime lying over p , we have

$$(47) \quad \rho|_{D_{0,\mathfrak{p}}} \cong \begin{pmatrix} \tilde{\epsilon}_{\mathfrak{p}} & * \\ 0 & \tilde{\delta}_{\mathfrak{p}} \end{pmatrix}, \text{ with } \tilde{\delta}_{\mathfrak{p}} \equiv \bar{\delta}_{\mathfrak{p}} \pmod{\mathfrak{m}_0} \text{ in } D_{0,\mathfrak{p}},$$

where \mathfrak{m}_0 is the maximal ideal of \mathcal{R}_0 .

Let $\rho_n := \rho|_{G_{E_n}}$. We also denote the unique prime of E_n lying above \mathfrak{p} by \mathfrak{p} . Let $D_{n,\mathfrak{p}}$ denote the decomposition group at the prime \mathfrak{p} . Then

$$(48) \quad \rho_n|_{D_{n,\mathfrak{p}}} \cong \begin{pmatrix} \tilde{\epsilon}_{n,\mathfrak{p}} & * \\ 0 & \tilde{\delta}_{n,\mathfrak{p}} \end{pmatrix}, \text{ with } \tilde{\delta}_{n,\mathfrak{p}} \equiv \bar{\delta}_{\mathfrak{p}} \pmod{\mathfrak{m}_n} \text{ in } D_{n,\mathfrak{p}}.$$

Let $\tilde{\delta}_{\infty,\mathfrak{p}}$ be the restriction of $\tilde{\delta}_{\mathfrak{p}}$ to $G_{E_{\infty,\mathfrak{p}}}$, and let Λ_∞ be the projective limit of the universal deformation rings for $\delta_{n,\mathfrak{p}} = \delta_{\mathfrak{p}}|_{E_{n,\mathfrak{p}}}$. Let $\tilde{\Lambda}_n$ be the subalgebra of \mathcal{R}_n topologically generated by the image of $\tilde{\delta}_{\infty,\mathfrak{p}}$ over \mathcal{O} . Assume that the order of $\tilde{\epsilon}_{n,\mathfrak{p}} \pmod{\mathfrak{m}_n}$ is prime to p . As $\tilde{\delta}_{\infty,\mathfrak{p}}$ restricted to the p -wild inertia subgroup factors through $\Gamma_{n,\mathfrak{p}}$, and the tame part has values in \mathcal{O} , therefore $\tilde{\Lambda}_n \cong \mathcal{O}[[\tilde{\delta}_{n,\mathfrak{p}}(\text{Frob}_{\mathfrak{p}}) - \delta_{\mathfrak{p}}(\text{Frob}_{\mathfrak{p}})]]$ inside \mathcal{R}_n , for the Frobenius element $\text{Frob}_{\mathfrak{p}}$ in $D_{\infty,\mathfrak{p}}$. Let $\tilde{\delta}_0(\text{Frob}_{\mathfrak{p}}) = a(\mathfrak{p}) \in \mathcal{R}_0$ and

$$(49) \quad \text{Jac}_A := \left(\det \left(\frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}} \right)_{\mathfrak{p}, \mathfrak{p}'|p} \pmod{\mathfrak{a}} \right) \in Q(A).$$

for any quotient integral domain $A = \mathcal{R}_0/\mathfrak{a}$ of characteristic 0 with quotient field $Q(A)$.

Proposition 5.5. *Let $\tilde{\Lambda}_0 = \mathcal{O}[[x_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ (under the normalization $\gamma_{\mathfrak{p}} \mapsto 1 + x_{\mathfrak{p}}$), and*

$$(50) \quad \text{Jac}_{\Lambda_0} := \det \left(\frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}} \right)_{\mathfrak{p}, \mathfrak{p}'|p} \in \tilde{\Lambda}_0^\times.$$

Then $\Lambda_\infty = \tilde{\Lambda}_0$.

Proof. Note that $\Omega_{\mathcal{O}[[x_{\mathfrak{p}}]]_{\mathfrak{p}|p}/\mathcal{O}} = \sum_{\mathfrak{p}|p} \tilde{\Lambda}_0 dx_{\mathfrak{p}}$. Consider the following exact sequence:

$$\Omega_{\Lambda_\infty/\mathcal{O}} \otimes \tilde{\Lambda}_0 \longrightarrow \Omega_{\tilde{\Lambda}_0/\mathcal{O}} \longrightarrow \Omega_{\tilde{\Lambda}_0/\Lambda_\infty} \longrightarrow 0.$$

The image of $\Omega_{\Lambda_\infty/\mathcal{O}} \otimes \tilde{\Lambda}_0$ is generated by $da(\mathfrak{p}) = \sum_{\mathfrak{p}'|p} \frac{\partial a(\mathfrak{p})}{\partial x_{\mathfrak{p}'}} \partial x_{\mathfrak{p}'}$. Therefore $\Omega_{\tilde{\Lambda}_0/\Lambda_\infty} \cong \frac{\sum_{\mathfrak{p}|p} \tilde{\Lambda}_0 dx_{\mathfrak{p}}}{\sum_{\mathfrak{p}|p} \tilde{\Lambda}_0 da(\mathfrak{p})}$. Since the Jacobian is a unit in $\tilde{\Lambda}_0$, we have $\Omega_{\tilde{\Lambda}_0/\Lambda_\infty} = 0$. Therefore $\tilde{\Lambda}_0 = \Lambda_\infty$. \square

For any number field L , let S_L be the set of primes of L lying above p and $D_L = \prod_{\mathfrak{p} \in S_L} D_{\mathfrak{p},p}^{ab}$, where $D_{\mathfrak{p},p}^{ab}$ is the maximal p -profinite abelian quotient of the decomposition subgroup $D_{\mathfrak{p}}$ at \mathfrak{p} in Gal_L . Let $I_L = \prod_{\mathfrak{p} \in S_L} I_{\mathfrak{p},p}^{ab}$, where $I_{\mathfrak{p},p}^{ab}$ is the inertia subgroup of $D_{\mathfrak{p},p}^{ab}$. Let E be a totally real field, and let f be a nearly p -ordinary Hilbert modular form, which is a Hecke eigenform. Let ρ be the representation of Gal_E that is associated to f . Let ρ be the nearly ordinary deformation for ρ . Let $\mathcal{R}_{E_n}^{n,ord}$ be the universal nearly ordinary deformation ring and $h_{E_n}^{n,ord}$ be the nearly ordinary Hecke algebra, for the Galois representation ρ restricted to Gal_{E_n} . Let $\text{Cl}_{E_n}(p^\infty)_p$ be the Galois group of the maximal p -profinite abelian extension of E_n unramified outside p and the Archimedean primes. Let \mathbb{I} be an irreducible component of $\mathcal{R}_E^{n,ord}$. In [Hid00a, Theorem 6.3], Hida gave the structure of Hecke algebras along the cyclotomic tower of a number field. We give the following generalization of Hida's theorem.

Theorem 5.6. *Let $s = |S_E|$ be the number of primes of E lying above p , and $J := \text{Jac}_{\mathbb{I}}$ be the Jacobian.*

(i) *Let $\text{Sel}_E(\text{Ad}^0(\rho)) = 0$ and $J \in \mathbb{I}^\times$. Then*

$$(51) \quad \begin{aligned} \mathcal{R}_{E_n}^{n,ord} &\cong h_{E_n}^{n,ord} \cong \mathcal{O}[[D_n \times \text{Cl}_{E_n}(p^\infty)_p]]; \\ \text{Sel}_{E_\infty}(\text{Ad}^0(\rho)) &\cong \mathbb{I}[S_E] \end{aligned}$$

(ii) *Let $M := \Omega_{\mathcal{R}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{\mathcal{R}_\infty} \mathbb{I}$. Then we have the following short-exact sequence:*

$$0 \longrightarrow \text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho)) \longrightarrow M \times (\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}) \longrightarrow \Omega_{\mathbb{I}/A_0} \otimes_{\mathbb{I}} \mathbb{I} \longrightarrow 0.$$

(iii) *Let $E_{n,cyc}$ be the cyclotomic \mathbb{Z}_p -extension of E_n , and $\Gamma_n := \text{Gal}(E_{n,cyc}/E)$. Then the module $\text{Sel}_{E_{n,cyc}}(\text{Ad}^0(\rho))$ is torsion over $\mathbb{I}[[\Gamma_n]] \cong \mathbb{I}[[T]]$ for all n ; and is pseudo-isomorphic to $\mathbb{I}^s \oplus \Omega_{\mathcal{R}_{E_{n,cyc}}/\mathcal{O}[[D_{cyc}]]} \otimes_{\mathbb{I}} \mathbb{I}$.*

(iv) *Let $\Phi(T)$ be the characteristic ideal of M , and $\Psi(T)$ the characteristic ideal of $\text{Sel}_{E_{n,cyc}}(\text{Ad}^0(\rho))$. Then*

$$(52) \quad \Psi(T) = \Phi(T)T^s, \Phi(0) \neq 0 \text{ and } \Phi(0) \mid J\eta,$$

where η is the characteristic ideal of the \mathbb{I} -module $\text{Sel}_{E_n}(\text{Ad}^0(\rho))$.

Proof. We give a proof for (ii), as the proof of the rest of the statements are in [Hid00a, Theorem 6.3]. Let $\mathcal{R}_j := \mathcal{R}_{E_j}^{\phi'}$, $A_j := \mathcal{O}[[D_j]]$ and $\Lambda := \mathcal{O}[[I_E]]$. Put $J_j := \ker(\mathcal{R}_j \rightarrow \mathbb{I})$. Then, we have the following commutative diagram with exact rows and columns, for all $j = 1, \dots, \infty$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{A_j/\mathcal{O}[[I_j]]} \otimes_{A_j} \mathbb{I} & \xlongequal{\quad} & \Omega_{A_j/\mathcal{O}[[I_j]]} \otimes_{A_j} \mathbb{I} & \longrightarrow & 0 \\ & & \downarrow e & & \downarrow f & & \\ \frac{J_j}{J_j^2} \otimes_{\mathbb{I}} \mathbb{I} & \longrightarrow & \Omega_{R_j/\mathcal{O}[[I_j]]} \otimes_{R_j} \mathbb{I} & \xrightarrow{b} & \Omega_{\mathbb{I}/\mathcal{O}[[I_j]]} \otimes_{R_j} \mathbb{I} & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow h & & \\ \frac{J_j}{J_j^2} \otimes_{\mathbb{I}} \mathbb{I} & \longrightarrow & \Omega_{R_j/A_j} \otimes_{R_j} \mathbb{I} & \xrightarrow{d} & \Omega_{\mathbb{I}/A_j} \otimes_{R_j} \mathbb{I} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Since the Jacobian $\text{Jac}_{\mathbb{I}} \neq 0$, the maps e and f are injective. Therefore, for $j = \infty$, we have the following short-exact sequence:

$$0 \longrightarrow \Omega_{A_j/\mathcal{O}[[I_j]]} \otimes_{A_j} \mathbb{I} \xrightarrow{\beta} M \times (\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}) \xrightarrow{\alpha} \Omega_{\mathbb{I}/A_0} \otimes_{\mathbb{I}} \mathbb{I} \longrightarrow 0,$$

where $\alpha(m, a) = d(m) - h(a)$ and $\beta(a) = (g(a), b(a))$. Again as the Jacobian vanishes, the modules $\Omega_{\mathbb{I}/A_0}$ and $\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]}/(\mathbb{I}[S_E])$ are torsion over \mathbb{I} . Finally, as $\text{Sel}_{F_\infty}^*(\text{Ad}^0(\rho)) \cong \Omega_{A_j/\mathcal{O}[[I_j]]} \otimes_{A_j} \mathbb{I}$, we have the following exact sequence:

$$0 \longrightarrow \text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho)) \xrightarrow{\beta} M \times (\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}) \xrightarrow{\alpha} \Omega_{\mathbb{I}/A_0} \otimes_{\mathbb{I}} \mathbb{I} \longrightarrow 0.$$

□

This result gives a finer structure of the dual Selmer group $\text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho))$ of the nearly ordinary representation ρ . Over the cyclotomic \mathbb{Z}_p -extension, Hida interprets the finer structure of $\text{Sel}_{E_{\text{cyc}}}^*(\text{Ad}^0(\rho))$ in terms of trivial zeros of the p -adic L-function. However, in the case when we have other p -adic Lie extensions, a more general interpretation seems to be needed (see §6.7 below).

6 Noncommutative Iwasawa theory of $\text{Sel}_{E_\infty}^*(\text{Ad}^0(\phi))$

6.1 Ore sets and the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$

Let E_∞/E be a p -adic Lie extension such that $E_{\text{cyc}} \subset E_\infty$. Let $\mathcal{G} := \text{Gal}(E_\infty/E)$ and $\mathcal{H} = \text{Gal}(E_\infty/E_{\text{cyc}})$. We will consider an analogue of the *Ore set*, that was first considered by Venjakob for a formulation of the Iwasawa Main conjecture over p -adic Lie extensions (see [CFKSV05]). We recall the Ore set that was considered by Venjakob.

Definition 6.1. Let \mathcal{O} be a finite extension of \mathbb{Z}_p . Then the set

$$(53) \quad S := \{x \in \mathcal{O}[[\mathcal{G}]] \mid \mathcal{O}[[\mathcal{G}]]/x \text{ is a finitely generated module over } \mathcal{O}[[\mathcal{H}]]\}.$$

is a left-right Ore set.

The following Ore set is a natural and obvious generalization of the one which has been considered by Venjakob, Coates et al in [CFKSV05] and in Fukaya-Kato [FK06]. Let \mathbb{I} be an irreducible component of the universal locally cyclotomic deformation ring \mathcal{R}_E for the functor Φ_E^{cyc} .

Definition 6.2. The set defined by

$$(54) \quad \mathcal{S} := \{x \in \mathbb{I}[[\mathcal{G}]] \mid \mathbb{I}[[\mathcal{G}]]/x \text{ is a finitely generated module over } \mathbb{I}[[\mathcal{H}]]\}$$

is a left-right Ore set.

Lemma 6.3. *The set \mathcal{S} is a multiplicatively closed set.*

Proof. For two elements $x, y \in \mathbb{I}[[\mathcal{G}]]$ consider the following exact sequence

$$(55) \quad 0 \longrightarrow x\mathbb{I}[[\mathcal{G}]]/xy \longrightarrow \mathbb{I}[[\mathcal{G}]]/xy \longrightarrow \mathbb{I}[[\mathcal{G}]]/x \longrightarrow 0.$$

The surjection $\mathbb{I}[[\mathcal{G}]]/y \longrightarrow x\mathbb{I}[[\mathcal{G}]]/xy \longrightarrow 0$ implies that $x\mathbb{I}[[\mathcal{G}]]/xy$ is finitely generated over $\mathbb{I}[[\mathcal{H}]]$ and the lemma follows. □

Definition 6.4. Let \mathfrak{m} denote the maximal ideal of \mathbb{I} . We define

$$(56) \quad \mathcal{S}^* := \cup_n \mathfrak{m}^n \mathcal{S}.$$

The set \mathcal{S}^* is also a multiplicative Ore set. In his thesis, Barth [Bar09] has also considered an Ore set which is different from ours.

Definition 6.5. We denote the category of all modules which are finitely generated over $\mathbb{I}[[\mathcal{G}]]$ and \mathcal{S}^* -torsion by $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$.

For the maximal ideal \mathfrak{m} of \mathbb{I} , we define

$$\begin{aligned} (57) \quad M[\mathfrak{m}] &:= \{x \in M \mid ax = 0 \text{ for some } a \in \mathfrak{m}\} \\ (58) \quad M(\mathfrak{m}) &:= \cup_n M[\mathfrak{m}^n]. \end{aligned}$$

As \mathbb{I} is an commutative integral domain, it is easy to see that $M[\mathfrak{m}]$ and $M(\mathfrak{m})$ are submodules of M over $\mathbb{I}[[\mathcal{G}]]$.

As in [CFKSV05, Lemma 2.1], we have the following characterization of the Ore set \mathcal{S} .

Lemma 6.6. *Let $\varphi_{\mathcal{H}} : \mathbb{I}[[\mathcal{G}]] \rightarrow \mathbb{I}[[\Gamma]]$ and $\psi_{\mathcal{H}} : \mathbb{I}[[\mathcal{G}]] \rightarrow \Omega(\Gamma)$ be the natural surjections. Then*

- (i) \mathcal{S} is the set of all x in $\mathbb{I}[[\mathcal{G}]]$ such that $\mathbb{I}[[\Gamma]]/\mathbb{I}[[\Gamma]]\varphi_{\mathcal{H}}(x)$ is a finitely generated \mathbb{I} -module;
- (ii) \mathcal{S} is the set of all x in $\mathbb{I}[[\mathcal{G}]]$ such that $\Omega(\Gamma)/\Omega(\Gamma)\psi_{\mathcal{H}}(x)$ is finite.

Proof. For any element $x \in \mathbb{I}[[\mathcal{G}]]$, we put $M = \mathbb{I}[[\mathcal{G}]]/\mathbb{I}[[\mathcal{G}]]x$. Then

$$(59) \quad M_{\mathcal{H}} = \mathbb{I}[[\Gamma]]/\mathbb{I}[[\Gamma]]\varphi_{\mathcal{H}}(x), \quad M/\mathfrak{m}_{\mathcal{H}}M = \Omega(\Gamma)/\Omega(\Gamma)\psi_{\mathcal{H}}(x),$$

where $\mathfrak{m}_{\mathcal{H}}$ denotes the maximal ideal of $\mathbb{I}[[\mathcal{H}]]$. Therefore the assertions follow from Nakayama's lemma. \square

Proposition 6.7. *A finitely generated module M over $\mathbb{I}[[\mathcal{G}]]$ is \mathcal{S} -torsion if and only if M is finitely generated over $\mathbb{I}[[\mathcal{H}]]$.*

Corollary 6.8. *A finitely generated module M over $\mathbb{I}[[\mathcal{G}]]$ is \mathcal{S}^* -torsion if and only if $M/M(\mathfrak{m})$ is finitely generated over $\mathbb{I}[[\mathcal{H}]]$.*

We now recall another way of describing the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$. Consider the canonical injection $i : \mathbb{I}[[\mathcal{G}]] \rightarrow \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$. First recall that $K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$ is an abelian group, whose group law is denoted additively. Consider triples (P, α, Q) , with P and Q finitely generated projective modules over $\mathbb{I}[[\mathcal{G}]]$ and α is an isomorphism between $P \otimes_{\mathbb{I}[[\mathcal{G}]]} \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$ and $Q \otimes_{\mathbb{I}[[\mathcal{G}]]} \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$ over $\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$. A morphism between (P, α, Q) and (P', α', Q') is naturally defined to be a pair of $\mathbb{I}[[\mathcal{G}]]$ -module homomorphism $g : P \rightarrow P'$ and $h : Q \rightarrow Q'$ such that

$$\alpha' \circ (\text{id}_{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}} \otimes g) = (\text{id}_{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}} \otimes h) \circ \alpha.$$

Note that it is an isomorphism if both g and h are isomorphisms. We denote the isomorphism class by $[(P, \alpha, Q)]$. Then the abelian group $K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$, is defined by the following generators and relations. Generators are the isomorphism classes $[(P, \alpha, Q)]$ and the relations are given by

- (i) $[(P, \alpha, Q)] = [(P', \alpha', Q')]$ if (P, α, Q) is isomorphic to (P', α', Q')
- (ii) $[(P, \alpha, Q)] = [(P', \alpha', Q')] + [(P'', \alpha'', Q'')]$
for every short exact sequence $0 \rightarrow [(P', \alpha', Q')] \rightarrow [(P, \alpha, Q)] \rightarrow [(P'', \alpha'', Q'')] \rightarrow 0$ in \mathcal{C}_i .
- (iii) $[(P_1, \beta \circ \alpha, P_3)] = [(P_1, \alpha, P_2)] + [(P_2, \alpha, P_3)]$, for the map $P_1 \xrightarrow{\alpha} P_2 \xrightarrow{\beta} P_3$.

Recall the category \mathcal{C}_i , whose objects are bounded complexes of finitely generated projective $\mathbb{I}[[\mathcal{G}]]$ -modules whose cohomologies are \mathcal{S} -torsion. Then the abelian group $K_0(\mathcal{C}_i)$ is defined with the following set of generators and relations. The generators are given by $[C]$, where C is an object of \mathcal{C}_i . The relations are given by

- (i) $[C] = 0$ if C is acyclic,
- (ii) $[C] = [C'] + [C'']$, for every short-exact sequence $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ in \mathcal{C}_i .

It is known that $K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \cong K_0(\mathcal{C}_i)$. Moreover, if $\mathcal{H}_{\mathcal{S}}$ is the category of all finitely generated $\mathbb{I}[[\mathcal{G}]]$ -modules which are \mathcal{S} -torsion and which have a finite resolution by finitely generated projective modules then $K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \cong K_0(\mathcal{H}_{\mathcal{S}})$. For details see Weibel [Wei07]. Therefore $K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$ is isomorphic to $K_0(\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G}))$. We then have the following exact sequence sequence of localization:

$$(60) \quad K_1(\mathbb{I}[[\mathcal{G}]]) \longrightarrow K_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \xrightarrow{\partial} K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \longrightarrow K_0(\mathbb{I}[[\mathcal{G}]]) \longrightarrow K_0(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}).$$

Regarding the connecting homomorphism ∂ , we have the following generalization of [Kak13, Lemma 5] and [CFKSV05, Prop 3.4].

Lemma 6.9. *The connecting homomorphism ∂ is surjective.*

Proof. We give only a brief sketch of the proof. Let P be a pro- p open normal subgroup of \mathcal{G} , and L be a finite extension of \mathbb{Q}_p such that all the irreducible representations of $\Delta = \mathcal{G}/P$ are defined. Then, we have an isomorphism of rings $L[\Delta] \xrightarrow{\cong} \prod_{\psi: \text{irred}} M_{n_{\psi}}(L)$, where ψ runs over all the irreducible representations of Δ and n_{ψ} is the dimension of ψ . Let $\mathbb{I} = \mathcal{O}[[X_1, \dots, X_r]]$ and $\mathbb{K} := L[[X_1, \dots, X_r]]$. Then tensoring with \mathbb{I} , we have

$$\mathbb{K}[\Delta] \xrightarrow{\cong} \prod_{\psi: \text{irred}} M_{n_{\psi}}(\mathbb{K}).$$

Then, there is a map $\lambda : K_0(\mathbb{I}[[\mathcal{G}]]) \longrightarrow \prod_{\psi: \text{irred}} K_0(\mathbb{K})$ which is constructed analogously as in Coates et. al [CFKSV05]. In fact, this map is constructed as the composition $\lambda = \lambda_4 \circ \lambda_3 \circ \lambda_2 \circ \lambda_1$ of the following natural maps

$$\begin{aligned} \lambda_1 &: K_0(\mathbb{I}[[\mathcal{G}]]) \longrightarrow K_0(\mathbb{I}[\Delta]), \\ \lambda_2 &: K_0(\mathbb{I}[\Delta]) \longrightarrow K_0(\mathbb{K}[\Delta]), \\ \lambda_3 &: K_0(\mathbb{K}[\Delta]) \longrightarrow K_0(\mathbb{K}[\Delta]), \\ \lambda_4 &: K_0(\mathbb{K}[\Delta]) \xrightarrow{\cong} \prod_{\psi: \text{irred}} K_0(M_{n_{\psi}}(\mathbb{K})) \xrightarrow{\cong} \prod_{\psi: \text{irred}} K_0(\mathbb{K}). \end{aligned}$$

The map λ_1 is defined analogously as in [CFKSV05, Lemma 3.5], and λ_2, λ_3 are induced by the inclusion of rings. The map λ_4 is induced by the isomorphism above followed by Morita equivalence. After this, the proof proceeds as in [Kak13, Lemma 5]. A crucial input here is a generalization of a result of Venjakob [Ven05], that if U is finitely generated \mathcal{S} -torsion over $\mathbb{I}[[\mathcal{G}]]$, then the twist $tw_{\psi}(U) := U \otimes_{\mathbb{I}} \mathbb{I}^{n_{\psi}}$, for any irreducible representation ψ of Δ , is also finitely generated and \mathcal{S} -torsion over $\mathbb{I}[[\mathcal{G}]]$. This also follows analogously as in *loc. cit.* \square

As a generalization of Conjecture 5.1 in [CFKSV05], we can hope that the following is true.

Conjecture 6.10. The dual Selmer group $\text{Sel}_{E_{\infty}}^*(\text{Ad}^0(\rho))$ is in the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$.

We can compare the following Ore set considered in [CFKSV05] with the multiplicative set \mathcal{S} ,

$$(61) \quad S = \{h \in \mathcal{O}[[\mathcal{G}]] \mid \mathcal{O}[[\mathcal{G}]]/h \text{ is finitely generated as a module over } \mathcal{O}[[\mathcal{H}]]\}.$$

Proposition 6.11. *Let $\phi_k : \mathbb{I} \longrightarrow \mathcal{O}$ be a specialization map. Then $\phi_k(\mathcal{S}) = S$.*

Proof. Let $x \in \mathcal{S}$. Then there exists a positive integer m , such that $\mathbb{I}(\mathcal{H})^m \twoheadrightarrow \mathbb{I}[[\mathcal{G}]]/x$. Applying ϕ_k , we get the following diagram

$$\begin{array}{ccccc} \mathbb{I}(\mathcal{H})^m & \longrightarrow & \mathbb{I}[[\mathcal{G}]]/x & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}[[\mathcal{H}]]^m & \longrightarrow & \mathcal{O}[[\mathcal{G}]]/\phi_k(x) & & \\ \downarrow & & \downarrow & & \\ 0 & & 0. & & \end{array}$$

Since the specialization map is surjective, the vertical maps induced by the specialization map ϕ_k are also surjective. Therefore $\phi_k(x) \in S$ ([CFKSV05, Lemma 2.1]).

Conversely, let $y \in S$. Then, we have a surjection $\mathcal{O}[[\mathcal{H}]]^m \rightarrow \mathcal{O}[[\mathcal{G}]]/y \rightarrow 0$ for some m . Since ϕ_k is surjective, there exists $z \in \mathbb{I}(\mathcal{H})$ such that $\phi_k(z) = y$. Further, $\mathcal{O}[[\mathcal{G}]] \cong \mathbb{I}[[\mathcal{G}]]/\ker \phi_k$. Therefore, $\mathcal{O}[[\mathcal{G}]]/y \cong \frac{\mathbb{I}[[\mathcal{G}]]/\ker \phi_k}{z} \cong \frac{\mathbb{I}[[\mathcal{G}]]/z}{\ker \phi_k}$, which is finitely generated over $\mathcal{O}[[\mathcal{H}]] \cong \mathbb{I}(\mathcal{H})/\ker \phi_k$. Therefore, $\frac{\mathbb{I}[[\mathcal{G}]]/z}{\mathfrak{n}}$ is finitely generated over $\mathbb{I}(\mathcal{H})/\mathfrak{n}$, where \mathfrak{n} is the maximal ideal of $\mathbb{I}(\mathcal{H})$. By Nakayama's lemma, $\mathbb{I}[[\mathcal{G}]]/z$ is finitely generated over $\mathbb{I}(\mathcal{H})$. Hence $z \in \mathcal{S}$. \square

Corollary 6.12. *For any specialization map ϕ_k , $\phi_k(\mathcal{S}^*) = S^*$.*

6.2 Noetherian Deformation rings and $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}(\mathcal{G})$

We now give some results which are extensions of the results over the cyclotomic \mathbb{Z}_p -extension [Hid06, Th 5.9, Cor 5.10, 5.11] to the p -adic Lie extension case.

Proposition 6.13. *Let P be a locally cyclotomic arithmetic point of weight k . Then*

$$(62) \quad \text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_P) \otimes_W W^*) \cong \Omega_{\mathcal{R}_{\infty}/W} \otimes_{\mathcal{R}_{\infty}} \mathcal{R}_0/P,$$

as $W[[\mathcal{G}]]$ -modules. Further, $\text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_P) \otimes_W W^*)$ is a $W[[\mathcal{G}]]$ -module of finite type. Here W^* is the Pontryagin dual of W .

Proof. Let $\pi_n : \mathcal{R}_n \rightarrow \mathcal{R}_0$ be the base change morphism. Let $P_n = \pi_n^{-1}(P)$ and consider the module $\Omega_{\mathcal{R}_n/W}$. Note that $\mathcal{R}_{\infty}/P_{\infty} = \mathcal{R}_n/P_n$, and

$$(63) \quad \text{Sel}_{F_n}^*(\text{Ad}^0(\rho_P) \otimes_W W^*) \cong \Omega_{\mathcal{R}_n/W} \otimes_{\mathcal{R}_{\infty}} \mathcal{R}_n/P_n$$

Taking projective limits, we have

$$(64) \quad \text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_P) \otimes_W W^*) \cong \Omega_{\mathcal{R}_{\infty}/W} \otimes_{\mathcal{R}_{\infty}} \mathcal{R}_0/P.$$

\square

Proposition 6.14. *If the p -adic Lie extension F_{∞} is totally ramified over F , and $e = |\Sigma_p|$ is the number of primes of F above p , then $\dim \mathcal{R}_{m,P} = e + 1$. Further, let P be a locally cyclotomic point over \mathcal{R}_n , which we may regard as a point over \mathcal{R}_{∞} . Then for any finite index subgroup Δ_m , with $m \geq n$, we have*

$$(65) \quad (\mathcal{R}_{\infty,P})_{\Delta_m} \cong \mathcal{R}_{n,P}.$$

Proof. Since $\mathcal{R}_{m,P}$ is an integral domain of dimension $e + 1$, and the base change morphism $\mathcal{R}_{m,P} \rightarrow \mathcal{R}_{n,P}$ of W -algebras is surjective, we have $\dim \mathcal{R}_{m,P} = \dim \mathcal{R}_{n,P}$. It follows that $\mathcal{R}_{m,P} \cong \mathcal{R}_{n,P}$, and the result follows. \square

Theorem 6.15. *Consider the representation $\rho_{\mathbb{I}} : \text{Gal}_F \rightarrow GL_2(\mathbb{I})$ and $\phi_k : \mathbb{I} \rightarrow \mathcal{O}$ be any surjective morphism of local algebras which give rise to a locally cyclotomic point P of weight k . The dual Selmer group $\text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_{\mathbb{I}}))$ is \mathcal{S} -torsion if and only if $\text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_P))$ is S -torsion.*

Proof. If $\text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_{\mathbb{I}}))$ is \mathcal{S} -torsion, then it is easy to see that $\text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_P))$ is S -torsion.

By Proposition 6.13, we have

$$(66) \quad \text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_P) \otimes_W W^*) \cong \Omega_{\mathcal{R}_{\infty}/W} \otimes_{\mathcal{R}_{\infty}} \mathcal{R}_0/P,$$

as $W[[\mathcal{G}]]$ -modules. Let $\mathcal{M} := \Omega_{\mathcal{R}_\infty/W} \otimes_{\mathcal{R}_\infty} \mathbb{I}$. Note that \mathcal{M} is a finitely generated $\mathbb{I}[[\mathcal{G}]]$ -module. Then under the specialization map $\phi_k : \mathbb{I} \rightarrow \mathcal{O}$ with $\ker(\phi_k) = P$, by the above isomorphism, the $\mathcal{O}[[\mathcal{G}]]$ -module $M := \mathcal{M} \otimes_{\mathcal{R}_0} \mathcal{O}/P$ is finitely generated. Let $\{y_1, \dots, y_m\}$ be a set of generators for M over $\mathcal{O}[[\mathcal{G}]]$. By Nakayama's Lemma, a lift of these generators to \mathcal{M} , say $\{z_1, \dots, z_m\}$ generates \mathcal{M} . We now have the following commutative diagram:

$$\begin{array}{ccccc} \bigoplus_{j=1}^m \mathbb{I}[[\mathcal{G}]]z_j & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \bigoplus_{j=1}^m \mathcal{O}[[\mathcal{G}]]y_j & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Let M be S -torsion. Then each y_j in M is annihilated by $s_j \in S$. Let t_j be a lift of s_j to $\mathbb{I}[[\mathcal{G}]]$, and $\alpha_j \in \mathbb{I}[[\mathcal{G}]]$ be an annihilator of $t_j z_j$. Consider $\alpha_j t_j \in \mathbb{I}[[\mathcal{G}]]$. Then $\alpha_j t_j \neq 0$ as $\mathbb{I}[[\mathcal{G}]]$ has no nonzero divisors. Let β_j be the image of $\alpha_j \in \mathcal{O}[[\mathcal{G}]]$. Then $\beta_j s_j$ annihilates y_j . We now have the commutative diagram:

$$\begin{array}{ccccc} \mathbb{I}[[\mathcal{G}]]/\alpha_j t_j & \longrightarrow & \mathbb{I}[[\mathcal{G}]]z_j & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}[[\mathcal{G}]]/\beta_j s_j & \longrightarrow & \mathcal{O}[[\mathcal{G}]]y_j & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

Note that the surjective map $\mathcal{O}[[\mathcal{G}]]/\beta_j s_j \rightarrow \mathcal{O}[[\mathcal{G}]]y_j$ factors through $\mathcal{O}[[\mathcal{G}]]/s_j$. Therefore, we have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{I}[[\mathcal{G}]]/\alpha_j t_j & \longrightarrow & \mathbb{I}[[\mathcal{G}]]z_j & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}[[\mathcal{G}]]/s_j & \longrightarrow & \mathcal{O}[[\mathcal{G}]]y_j & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

Let $x = \alpha_j t_j$ and $s = s_j$. As $P = \ker(\mathbb{I} \rightarrow \mathcal{O})$, we have $\mathcal{O}[[\mathcal{G}]] \cong \mathbb{I}[[\mathcal{G}]]/P$. Hence $\mathcal{O}[[\mathcal{G}]]/s \cong \mathbb{I}[[\mathcal{G}]]/P$. Further $\frac{\mathbb{I}[[\mathcal{G}]]/P}{x} \cong \frac{\mathbb{I}[[\mathcal{G}]]}{\langle x, P \rangle} \cong \frac{\mathbb{I}[[\mathcal{G}]]/x}{P}$. So $\frac{\mathbb{I}[[\mathcal{G}]]/x}{P}$ is finitely generated over $\mathcal{O}[[\mathcal{H}]] \cong \mathbb{I}[[\mathcal{H}]]/P$. Let \mathfrak{n} denote the maximal ideal of $\mathbb{I}[[\mathcal{H}]]$. Then $\frac{\mathbb{I}[[\mathcal{G}]]/x}{\mathfrak{n}}$ is finitely generated over $\mathbb{I}[[\mathcal{H}]]/\mathfrak{n}$. By Nakayama's Lemma, $\mathbb{I}[[\mathcal{G}]]/x$ is finitely generated over $\mathbb{I}[[\mathcal{H}]]$. It follows that each summand $\mathbb{I}[[\mathcal{G}]]z_j$ is finitely generated over $\mathbb{I}(\mathcal{H})$. Therefore, \mathcal{M} is finitely generated over $\mathbb{I}(\mathcal{H})$ and hence \mathcal{S} -torsion. \square

Now suppose that M is in the category $\mathfrak{M}_{\mathcal{H}}(\mathcal{G})$. Then $M/M(p)$ is S -torsion. The natural surjection $\mathcal{M} \rightarrow M/M(p)$ factors through the submodule $\mathcal{M}(\mathfrak{m})$ of \mathcal{M} . By a similar argument as in the above proof, we can see that $\mathcal{M}/\mathcal{M}(\mathfrak{m})$ is annihilated by \mathcal{S} . Therefore, \mathcal{M} is in the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}[[\mathcal{G}]]$. We therefore have the following consequence:

Theorem 6.16. *Consider the representation $\rho_{\mathbb{I}} : \text{Gal}_F \rightarrow GL_2(\mathbb{I})$ and $\phi_k : \mathbb{I} \rightarrow \mathcal{O}$ be any surjective morphism of local algebras which give rise to a locally cyclotomic point P of weight k . The dual Selmer group $\text{Sel}_{F_\infty}^*(\text{Ad}^0(\rho_{\mathbb{I}}))$ is \mathcal{S}^* -torsion if and only if $\text{Sel}_{F_\infty}^*(\text{Ad}^0(\rho_P))$ is S^* -torsion.*

6.3 Noetherian property of \mathcal{R}_∞

Proposition 6.17. *Let F_∞ be a p -adic Lie extension of a totally real field F such that $\mathcal{G} := \text{Gal}(F_\infty/F)$ is a p -adic Lie group of dimension two. Let $\mathcal{H} := \text{Gal}(F_\infty/F_{\text{cyc}})$ and $\Gamma := \text{Gal}(F_{\text{cyc}}/F)$. Then*

- (i) *the dual Selmer group $\Omega_{\mathcal{R}_\infty/W} \otimes W$ is a finitely generated module over $W[[\mathcal{H}]]$,*
- (ii) *the ring \mathcal{R}_∞ is not noetherian.*

Proof. (i) By the control theorem, we have $(\Omega_{\mathcal{R}_\infty/W} \otimes W)_\mathcal{H} \cong \Omega_{\mathcal{R}_{\text{cyc}}/W} \otimes W$. Suppose \mathcal{R}_∞ is a noetherian ring. Then \mathcal{R}_{cyc} is also noetherian. It follows that $\Omega_{\mathcal{R}_{\text{cyc}}/W} \otimes W$ is a finitely generated W -module. By Nakayama Lemma, $\Omega_{\mathcal{R}_\infty/W} \otimes W$ is a finitely generated module over $W[[\mathcal{H}]]$.

- (ii) Suppose \mathcal{R}_∞ is noetherian. Then the module $\Omega_{\mathcal{R}_\infty/W} \otimes \mathbb{F}$ is finite. By Nakayama Lemma, $\Omega_{\mathcal{R}_\infty/W} \otimes W$ is a finitely generated torsion module over $W[[\mathcal{H}]]$. Moreover, its μ -invariant over $W[[\mathcal{H}]]$ is zero. The fact that $\Omega_{\mathcal{R}_\infty/W} \otimes W$ is $W[[\mathcal{H}]]$ -torsion implies that $(\Omega_{\mathcal{R}_\infty/W} \otimes W)_\mathcal{H}$ is finitely generated and torsion over W , i.e., finite. Therefore, $\Omega_{\mathcal{R}_{\text{cyc}}/W} \otimes W$ is also finite. By [Hid00a, Theorem 6.3 (4)], the dual Selmer group $\Omega_{\mathcal{R}_{\text{cyc}}/W} \otimes W$ has no non-trivial finite submodules over $W[[\Gamma]]$. Therefore $\Omega_{\mathcal{R}_{\text{cyc}}/W} \otimes W = 0$, which is not possible. It follows that the ring \mathcal{R}_∞ is not noetherian. \square

Remark 6.18. Unlike the situation in the case of the cyclotomic \mathbb{Z}_p -extension over a totally real field, the ring \mathcal{R}_∞ is not noetherian over any p -adic Lie extension that contains the cyclotomic \mathbb{Z}_p -extension and of higher dimension. It was felt that the noetherian property of \mathcal{R}_∞ could be used to check the conjecture on the category $\mathfrak{M}_\mathcal{H}(\mathcal{G})$. However, this hope turned out to be an impossibility at least over totally real fields, where the dual Selmer group does not admit any pseudo-null submodules. A similar result might also hold for uniform pro- p groups which have no elements of finite order.

6.4 Periods of adjoint Galois representations

We briefly recall the periods of the representation $\text{Ad}^0(\rho)$ where ρ is the representation of Gal_F that is associated to the Hilbert modular form $f \in S_\kappa(\mathfrak{N}, \varepsilon; W)$. Let M_f denote the motive associated to the Hilbert modular cusp form f over E . Let $c_\infty^\pm(M_f)$ and $c_p^\pm(M_f)$ denote the Deligne periods and the p -adic periods of M_f . Then, by [Hid96, Theorems 5.2.1(ii), 5.2.2], since F is totally real, we have

$$c_p^\pm(\text{Ad}^0(M_f)(1)) = c_p^+(M_f(1))c_p^-(M_f)\delta_p(M_f(1)).$$

Let ψ be any Artin representation of the Galois group Gal_E . Then $\text{Ad}^0(\rho_f) \otimes \psi$ is also critical at 0, 1. Let d_ψ denote the dimension of ψ and d_\pm be the dimension of the \pm -eigenspaces of the action of complex conjugation on ψ . Then

$$c_p^\pm(\text{Ad}^0(M_f) \otimes \psi(1)) = (2\pi i c_p^\pm(\text{Ad}^0(M_f)(1)))^{d_\psi}.$$

It is conjectured in [Del79] that

$$(67) \quad \frac{L(\text{Ad}^0(\rho_f)(1), \psi, 0)}{(2\pi i c_p^\pm(\text{Ad}^0(M_f)(1)))^{d_\psi}} \in \bar{\mathbb{Q}}.$$

Here, we recall that the L-function $L(\text{Ad}^0(\rho_f)(1) \otimes \psi, s)$ is defined to be the Euler product defined as reciprocal of the product of the following polynomials:

$$(68) \quad P_{\mathfrak{q}}(\text{Ad}^0(\rho_f), \psi, T) := \det(1 - \text{Frob}_{\mathfrak{q}}^{-1}T \mid (\text{Ad}^0(\rho_f) \otimes \psi)^{I_{\mathfrak{q}}}) \in \mathcal{O}[T], \mathfrak{q} \neq \mathfrak{p};$$

$$(69) \quad P_{\mathfrak{p}}(\text{Ad}^0(\rho_f), \psi, T) := \det(1 - \text{Frob}_{\mathfrak{p}}^{-1}T \mid (\text{Ad}^0(\rho_f) \otimes \psi)^{I_{\mathfrak{p}}}) \in \mathcal{O}[T];$$

$$(70) \quad P_{\mathfrak{p}}(\mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho_f), \psi, T) := \det(1 - \text{Frob}_{\mathfrak{p}}^{-1}T \mid ((\mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho_f)) \otimes \psi)^{I_{\mathfrak{p}}}) \in \mathcal{O}[T];$$

$$(71) \quad P_{\mathfrak{p}}((\mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho_f))^*, \psi, T) := \det(1 - \text{Frob}_{\mathfrak{p}}^{-1}T \mid ((\mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho_f))^* \otimes \psi)^{I_{\mathfrak{p}}}) \in \mathcal{O}[T],$$

where $(\mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho_f))^*$ denotes the contragredient representation of $\mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho_f)$.

6.5 Non-commutative Main conjecture

The noncommutative Main conjecture of Iwasawa theory predicts that there is an element in the group $K_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}^*})$ such that its image under the connecting homomorphism of K -theory gives rise to the class of the dual Selmer group. More precisely, consider the connecting homomorphism

$$(72) \quad K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \xrightarrow{\tilde{\partial}} K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}).$$

Let $\phi_{\kappa} : \mathbb{I} \longrightarrow \mathcal{O}$ be any specialization map. Then this induces the following map

$$(73) \quad \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}} \xrightarrow{\phi_{\kappa}} \mathcal{O}[[\mathcal{G}]]_S,$$

where S is the multiplicative set in $\mathcal{O}[[\mathcal{G}]]$. This induces the homomorphisms in the following commutative diagram

$$(74) \quad \begin{array}{ccc} K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) & \xrightarrow{\tilde{\partial}} & K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \\ \tilde{\phi}_{\kappa} \downarrow & & \downarrow \\ K'_1(\mathcal{O}[[\mathcal{G}]]_S) & \xrightarrow{\partial} & K_0(\mathcal{O}[[\mathcal{G}]], \mathcal{O}[[\mathcal{G}]]_S). \end{array}$$

Now let ρ be any Artin representation of \mathcal{G} , say $\rho : \mathcal{G} \longrightarrow GL_n(\mathcal{O}')$. Then this induces the following homomorphism of rings

$$(75) \quad \rho : \mathcal{O}[[\mathcal{G}]] \longrightarrow M_n(\mathcal{O}''[[\Gamma]]),$$

for some finite extension \mathcal{O}'' of \mathcal{O} and \mathcal{O}' . Further, we have the following homomorphism

$$(76) \quad \Phi_{\rho} : \mathcal{O}[[\mathcal{G}]]_S \longrightarrow M_n(Q_{\mathcal{O}''}(\Gamma)),$$

where $Q_{\mathcal{O}''}(\Gamma)$ is the quotient field of $\mathcal{O}''[[\Gamma]]$. Therefore, we have

$$(77) \quad K'_1(\mathcal{O}[[\mathcal{G}]]_S) \longrightarrow K_1(M_n(Q_{\mathcal{O}''}(\Gamma))) \cong Q_{\mathcal{O}''}(\Gamma)^{\times}.$$

Now, let φ be the augmentation map $\mathcal{O}[[\mathcal{G}]]$ to \mathcal{O} , and \mathfrak{p} be the kernel of this map. Then the map φ can be extended to the map $\varphi' : Q_{\mathcal{O}''}(\Gamma) \longrightarrow L \cup \{\infty\}$, for some finite extension L of \mathbb{Q}_p and by putting $\varphi(x) = \infty$, if $x \notin \mathcal{O}[[\mathcal{G}]]_{\mathfrak{p}}$. The composition of the map $\tilde{\phi}_{\kappa}$ in the above commutative diagram with the map φ' gives us a map

$$(78) \quad \begin{array}{ccc} K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) & \longrightarrow & L \cup \{\infty\} \\ & & x \mapsto x(\rho). \end{array}$$

This map satisfies the following properties:

- (i) Let \mathcal{G}' be an open subgroup of \mathcal{G} . Let χ be a one dimensional representation of \mathcal{G}' and $\rho = \text{Ind}_{\mathcal{G}'}^{\mathcal{G}} \chi$. Consider the norm map $K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}^*}) \longrightarrow K'_1(\mathbb{I}[[\mathcal{G}']]_{\mathcal{S}})$. Then for any $\tilde{x} \in K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}^*})$, we have

$$\tilde{x}(\rho) = N(\tilde{x})(\chi).$$

- (ii) Let $\rho_1 : \mathcal{G} \longrightarrow GL_{n_1}(L)$ and $\rho_2 : \mathcal{G} \longrightarrow GL_{n_2}(L)$ be two Artin representations for some field extension L of \mathbb{Q}_p . Then for any $\tilde{x} \in K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}^*})$, we have

$$\tilde{x}(\rho_1 \oplus \rho_2) = \tilde{x}(\rho_1)\tilde{x}(\rho_2).$$

(iii) Let U be a subgroup of \mathcal{H} which is normal in \mathcal{G} . Then the homomorphism $\mathcal{G} \rightarrow \mathcal{G}/U$ induces the homomorphism $\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}} \rightarrow \mathbb{I}[[\mathcal{G}/U]]_{\mathcal{S}}$. Further, we get the homomorphism $\pi : K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \rightarrow K'_1(\mathbb{I}[[\mathcal{G}/U]]_{\mathcal{S}})$. Let $\rho : \mathcal{G}/U \rightarrow GL_n(L)$. Then we get an Artin representation $\text{inf}(\rho) : \mathcal{G} \rightarrow \mathcal{G}/U \rightarrow GL_n(L)$. For any $\tilde{x} \in K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$, we have

$$(79) \quad \tilde{x}(\text{inf}(\rho)) = \pi(\tilde{x})(\rho).$$

From the localization sequence (60), we get the following exact sequence

$$K'_1(\mathbb{I}[[\mathcal{G}]]) \rightarrow K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \xrightarrow{\partial} K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \rightarrow 0.$$

Conjecture 6.19. (Main Conjecture over $\mathbb{I}[[\mathcal{G}]]$) Let $f \in S_{\kappa}(\mathfrak{Np}, \varepsilon; \mathcal{O})$ be obtained through the arithmetic specialization $\phi_{\kappa} : \mathbb{I} \rightarrow \mathcal{O}$, for some finite extension \mathcal{O} of \mathbb{Z}_p . Let ρ_f be the representation of Gal_E that is associated to f , $V := \text{Ad}^0(\rho_f)$, and $V_{\mathfrak{p}}^+ := \mathcal{F}_{\mathfrak{p}}^+ \text{Ad}^0(\rho_f)$. Then there exists an element $\tilde{\xi} \in K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$ such that $\partial(\tilde{\xi}) = -[\text{Sel}_{E_{\infty}}^*(\text{Ad}^0(\rho))]$. Further, under the map in (78), the following interpolation properties are satisfied for all Artin representations α of \mathcal{G} with degree d_{α}

$$(80) \quad \tilde{\phi}_{\kappa}(\tilde{\xi})(\alpha) = \frac{L_{\Sigma}(V(1), \alpha, 0)}{(2\pi i c_p^{\pm}(\text{Ad}^0(M_f)(1)))^{d_{\alpha}}} \times \left[\frac{P_{\mathfrak{p}}(V \otimes \psi, T)}{P_{\mathfrak{p}}(V_{\mathfrak{p}}^+ \otimes \psi, T)} \right]_{T=1} P_{\mathfrak{p}}((V_{\mathfrak{p}}^+ \otimes \psi)^*, 1) \prod_{\mathfrak{q} | \mathfrak{N}, \mathfrak{q} \neq \mathfrak{p}} P_{\mathfrak{q}}(V \otimes \psi, 1).$$

Here $L_{\Sigma}(V(1), \alpha, 0)$ is the value of the L-function for the twisted adjoint representation with Euler factors for primes in the set $\Sigma := \{\mathfrak{q} \mid \mathfrak{Np}\}$ removed.

Remark 6.20. A similar Main conjecture is formulated in the thesis of Barth [Bar09]. Similar main conjectures can be formulated for any p -adic family of nearly ordinary Galois representations. See sections 4.3 and 4.4 for a discussion about the interpolation properties. See [FK06, Theorem 4.1.12] for the interpolation property for motives.

6.6 Main conjecture in the abelian case

Let F be a totally real field, and f be a Hilbert modular Hecke eigenform of weight κ , level \mathfrak{N} . We also assume that f is ordinary at all the primes above the prime p . We denote the Galois representation associated to f by ρ_f . Let ρ be the universal ordinary deformation of ρ_f . Consider the field F_{∞} which is the maximal abelian pro- p extension of F which is unramified outside the $p\mathfrak{N}$. Let $\mathcal{G} = \text{Gal}(F_{\infty}/F)$. Then $\mathcal{G} = \mathcal{H} \times \Gamma$ where Γ is the Galois group of the cyclotomic \mathbb{Z}_p -extension of F , and we assume that \mathcal{H} is some finite group. In fact, if Leopoldt conjecture for F is true, then \mathcal{H} will be a finite group. Then for $\mathbb{I} = \mathcal{O}[[X_1, \dots, X_r]]$, we get $\mathbb{I}[[\mathcal{G}]] \cong \mathbb{I}[[\mathcal{H}]][[\Gamma]]$.

In this context, the p -adic L-function of $\text{Ad}^0(\rho_f)$ over the quotient field of $\mathbb{I}[[\mathcal{G}]]$ is constructed by Hida and Tilouine in the case $F = \mathbb{Q}$. If F is totally real and its ring of integers has class number one, then the p -adic L-function has been constructed by Hsin-Tai Wu ([Wu01]). In general, it has been constructed by Rosso in [Ros15, Theorem 7.2]. The Main conjecture over the field \mathbb{Q} is also proven for the Selmer group of the adjoint in [Urb01] in some cases and over totally real fields F in certain cases in [Ros15, Section 10].

Then, we have $\mathcal{L}_p(X_1, \dots, X_r, \epsilon, T) = G(X_1, \dots, X_r, \epsilon, T)$, as ideals, where G is the characteristic ideal of the dual selmer group of $\text{Ad}^0(\rho)$. Here ϵ is any finite order character of \mathcal{G} . We now interpret this equality of the Iwasawa Main conjecture in terms of K -theory. In this situation, the category $\mathfrak{M}_{\mathcal{H}}^{\mathbb{I}}[[\mathcal{G}]]$ consists of all finitely generated modules which are \mathcal{S}^* -torsion.

Lemma 6.21. Suppose that the μ -invariant of the Selmer group $\text{Sel}_{F_{\text{cyc}}}^*(\text{Ad}^0(\rho_f))$ be equal to zero. Then $\text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho_f))$ is S -torsion. It follows that $\text{Sel}_{F_{\infty}}^*(\text{Ad}^0(\rho))$ is \mathcal{S} -torsion, and the p -adic L-function \mathcal{L}_p is a unit in $\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$.

Proof. Let $\mathcal{H} = \mathcal{H}' \times \mathcal{H}_p$, where \mathcal{H}_p is the p -part of \mathcal{H} and \mathcal{H}' the group whose order is prime-to- p . By the proof of the Main conjecture due to Urban and Rosso, mentioned above, we have $\mathcal{L}_p(X_1, \dots, X_r, \epsilon, T) = G(X_1, \dots, X_r, \epsilon, T)$, for every character ϵ of \mathcal{H}' . Furthermore, as $\text{Sel}_{F_\infty}^*(\text{Ad}^0(\rho))$ is \mathcal{S} -torsion, therefore Note that $\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$ is the localization at the prime ideal $\mathfrak{m}_{\mathbb{I}}$. Then we have the following decomposition

$$\mathbb{I}[[\mathcal{G}]]_{\mathfrak{m}_{\mathbb{I}}} \xrightarrow{\cong} \mathbb{I}[H' \times H_p][[T]]_{\mathfrak{m}_{\mathbb{I}}} \xrightarrow{\cong} \oplus_{\psi \in \widehat{H'}} \mathbb{I}[H_p][[T]]_{\mathfrak{m}_{\mathbb{I}}}.$$

It is enough for us to show that the image in each summand is a unit. Note that the image in each summand is $\mathcal{L}_p(X_1, \dots, X_r, \psi, T) = G(X_1, \dots, X_r, \psi, T)$. Since the μ -invariant is equal to zero, $G(X_1, \dots, X_r, \psi, T) \in \mathcal{S}$, and hence $G(X_1, \dots, X_r, \psi, T)$ is a unit in $\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$. \square

Now, let Y be any finitely generated $\mathbb{I}[[T]]$ -module which is annihilated by an element outside the maximal ideal $\mathfrak{m}_{\mathbb{I}}$ of \mathbb{I} . Then the characteristic ideal \mathcal{P} of Y belongs to $\mathbb{I}[[T]]_{\mathfrak{m}_{\mathbb{I}}}^\times$. Now consider the class in $K_0(\mathbb{I}[[T]], \mathbb{I}[[T]]_{\mathfrak{m}_{\mathbb{I}}})$ which is given by $\left[\left(Y, 0, \frac{\mathbb{I}[[T]]}{f\mathbb{I}[[T]]} \right) \right]$. Then, under the connecting homomorphism

$$\partial : K_1(\mathbb{I}[[T]]_{\mathfrak{m}_{\mathbb{I}}}) \cong \mathbb{I}[[T]]_{\mathfrak{m}_{\mathbb{I}}}^\times \longrightarrow K_0(\mathbb{I}[[T]], \mathbb{I}[[T]]_{\mathfrak{m}_{\mathbb{I}}}),$$

we have $\partial(\mathcal{P}) = \left[\left(Y, 0, \frac{\mathbb{I}[[T]]}{f\mathbb{I}[[T]]} \right) \right]$. Taking Y to be the module $\text{Sel}_{F_\infty}^*(\text{Ad}^0(\rho))$ we have the following theorem.

Theorem 6.22. *Let $\text{Sel}_{F_\infty}^*(\text{Ad}^0(\rho))$ be annihilated by an element of $\mathbb{I}[[\mathcal{G}]]$ outside the maximal ideal $\mathfrak{m}_{\mathbb{I}}$ of \mathbb{I} . Then under the connecting homomorphism $\partial : K_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \longrightarrow K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathfrak{m}_{\mathbb{I}}})$, the p -adic L -function is mapped to the class $-\text{Sel}_{F_\infty/F}^*(\text{Ad}^0(\rho))$.*

6.7 Remark on zeros

As a consequence of Theorem 5.6, we get the following result:

Proposition 6.23. *Recall the notations from Theorem 5.6. Then we have the following equality in $K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$:*

$$[\text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho))] = [\Omega_{\mathcal{R}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{\mathcal{R}_\infty} \mathbb{I}] + [\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}].$$

Assuming the noncommutative Main Conjecture, the p -adic L -function for $[\text{Sel}_{E_\infty}^(\text{Ad}^0(\rho))]$ arises as the product of the p -adic L -function for $[\Omega_{\mathcal{R}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{\mathcal{R}_\infty} \mathbb{I}]$ and $[\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}]$.*

Proof. By Theorem 5.6, we have the following short-exact sequence:

$$0 \longrightarrow \text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho)) \longrightarrow (\Omega_{\mathcal{R}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{\mathcal{R}_\infty} \mathbb{I}) \oplus (\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}) \longrightarrow \Omega_{\mathbb{I}/A_0} \otimes_{\mathbb{I}} \mathbb{I} \longrightarrow 0.$$

Then, we have the following equality in $K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$:

$$[\text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho))] + [\Omega_{\mathbb{I}/A_0} \otimes_{\mathbb{I}} \mathbb{I}] = [\Omega_{\mathcal{R}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{\mathcal{R}_\infty} \mathbb{I}] + [\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}].$$

Since the module $\Omega_{\mathbb{I}/A_0}$ is \mathbb{I} -torsion, therefore it is the trivial class, and we have,

$$[\text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho))] = [\Omega_{\mathcal{R}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{\mathcal{R}_\infty} \mathbb{I}] + [\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}].$$

Assuming the noncommutative Main Conjecture for the classes $[\Omega_{\mathcal{R}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{\mathcal{R}_\infty} \mathbb{I}]$ and $[\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}]$ there exists elements $\tilde{\psi}, \tilde{\tau} \in K_1'(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$ such that their images under the connecting homomorphism ∂ are the modules $[\Omega_{\mathcal{R}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{\mathcal{R}_\infty} \mathbb{I}]$ and $[\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}]$ in $K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$. Therefore, ∂ maps the product $\tilde{\psi}\tilde{\tau}$ to the class $[\text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho))]$. \square

Remark 6.24. Over the cyclotomic \mathbb{Z}_p -extension, we saw in Theorem 5.6 that the $\mathbb{I}[[T]]$ -module $\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]}$ is pseudo-isomorphic to T^s , where $s = S_F$ is the number of primes of E above p . In the case of a p -adic Lie extension also, the pre-image of the class $[\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{I}]$ occurs as a factor of the p -adic L -function of the class $[\text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho))]$.

7 K_1 computations and congruences over $\mathbb{I}[[\mathcal{G}]]$

In this section, we extend the strategy that has so far been followed to prove the Main conjecture. This strategy relying on a description of the K -groups was first used by Burns and then by Kato, who used it to prove certain instances of the Main conjecture over number fields. Independently, Ritter and Weiss also proved instances of the Main conjecture. Their ideas are similar to that of Burns and Kato. Kakde and Hara also proved instances of the Main conjecture for certain p -adic Lie extensions. They were inspired by the work of Burns and Kato. We extend their results suitably and show that the Main conjecture that we have formulated can also be established for certain p -adic families of Galois representations.

7.1 General strategy

The strategy involves reducing the proof of the Main conjecture over compact p -adic Lie groups to compact p -adic Lie groups of dimension one. For this, it is crucial to know that the completed group ring $\mathbb{I}[[\mathcal{G}]]$ is an adic ring. Indeed, if $\mathfrak{m}_{\mathbb{I}}$ is the maximal ideal of \mathbb{I} and $I_{\mathcal{G}}$ the augmentation ideal of $\mathbb{I}[[\mathcal{G}]]$, then $J_{\mathcal{G}} = \mathfrak{m}_{\mathbb{I}} + I_{\mathcal{G}}$ is the maximal ideal of $\mathbb{I}[[\mathcal{G}]]$, and $\mathbb{I}[[\mathcal{G}]]$ is an adic ring with respect to the ideals $\{J_{\mathcal{G}}^n : n \in \mathbb{N}\}$ in the sense of Fukaya-Kato ([FK06, 1.4.1]). Then it is shown in Fukaya-Kato ([FK06, Prop 1.5.1]) that

$$K_1(\mathbb{I}[[\mathcal{G}]]) \xrightarrow{\cong} \varprojlim_n K_1(\mathbb{I}[[\mathcal{G}]]/J_{\mathcal{G}}^n).$$

Following [Bur15], in [Kak13, §4], a series of reduction steps are made showing that the proof of the Main Conjecture for any arbitrary p -adic Lie group can be reduced to the case when the Galois group \mathcal{G} has dimension one, with $\mathcal{G} \cong \Delta \times \mathcal{G}_p$, where Δ is a finite cyclic group of order prime to p and \mathcal{G}_p is a pro- p compact p -adic Lie group of dimension one. We do not give the steps leading to this reduction, though the idea essentially is derived from the isomorphism (7.1) above. We proceed with the belief that similar reductions are possible.

Consider the Iwasawa algebra $\mathbb{I} \cong \mathcal{O}[[X_1, \dots, X_r]]$, for some r . Then $\mathbb{I}[[\mathcal{G}]] \cong \prod_{\psi \in \Delta^*} \mathbb{I}[\psi](\mathcal{G}_p)$, where $\mathbb{I}[\psi]$ is the algebra obtained by adjoining the values of ψ to \mathbb{I} . This further allows one to reduce the proof of the Main conjecture to the case when \mathcal{G} is a pro- p , compact p -adic Lie group of dimension one.

We now assume that \mathcal{G} is a p -adic Lie group of dimension 1. Let $\Sigma(\mathcal{G})$ be any set of rank 1 subquotients of \mathcal{G} of the form U^{ab} with U an open subgroup of \mathcal{G} that has the following property:

- (*) For each Artin representation ρ of \mathcal{G} , there is a finite subset $\{U_i^{ab} : i \in I\}$ of $\Sigma(\mathcal{G})$ and for each index i an integer m_i and a degree one representation ρ_i of U_i^{ab} such that there is an isomorphism of virtual representations $\rho \cong \sum_{i \in I} m_i \cdot \text{Ind}_{U_i}^{\mathcal{G}} \text{Ind}_{U_i^{ab}}^{U_i} \rho_i$.

Let U^{ab} be a subquotient satisfying the above property (*). Note that we have the following natural homomorphism,

$$(81) \quad K_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) \longrightarrow K_1(\mathbb{I}(U)_{\mathcal{S}}) \longrightarrow K_1(\mathbb{I}(U^{ab})_{\mathcal{S}}) \longrightarrow \mathbb{I}(U^{ab})_{\mathcal{S}}^{\times} \subset Q_{\mathbb{I}}(U^{ab})^{\times}.$$

Taking all the U^{ab} in $\Sigma(\mathcal{G})$ we get the following homomorphism

$$(82) \quad \Theta_{\Sigma(\mathcal{G})} : K_1(\mathbb{I}[[\mathcal{G}]]) \longrightarrow \prod_{U^{ab} \in \Sigma(\mathcal{G})} Q_{\mathbb{I}}(U^{ab})^{\times}.$$

Definition 7.1. Let $\mathbb{K} = K[[X_1, \dots, X_r]]$, where K is the quotient field of \mathcal{O} , and Y is the variable corresponding to $\tilde{\Gamma}^{p^e}$. For any finite group G , we consider the following groups (see [Oli88, Page 173]):

$$\begin{aligned} SK_1(\mathbb{I}[[\mathcal{G}]]) &:= \ker [K_1(\mathbb{I}[[\mathcal{G}]]) \longrightarrow K_1(\mathbb{K}[[\mathcal{G}]])], \\ K'_1(\mathbb{I}[[\mathcal{G}]]) &:= K_1(\mathbb{I}[[\mathcal{G}]]) / SK_1(\mathbb{I}[[\mathcal{G}]]) , \end{aligned}$$

where μ_K is the set of roots of unity in K .

Proposition 7.2. *Let the μ -invariant of the dual Selmer group $\text{Sel}_{E_{\text{cyc}}}^*(\text{Ad}^0(\rho))$ be equal to zero. Then the Main Conjecture in 6.19 is valid if and only if for any set of subquotients $\Sigma(\mathcal{G})$ with the property (*) above, the following two conditions hold:*

- (i) *there exists a subgroup Φ of $\prod_{U^{ab} \in \Sigma(\mathcal{G})} \mathbb{I}(U^{ab})^\times$ such that $\Theta_{\Sigma(\mathcal{G})} : K_1(\mathbb{I}[[\mathcal{G}]]) \rightarrow \Phi$ is an isomorphism;*
- (ii) *there exists a subgroup $\Phi_{\mathcal{S}}$ of $\prod_{U^{ab} \in \Sigma(\mathcal{G})} \mathbb{I}(U^{ab})_{\mathcal{S}}^\times$ such that $\Phi_{\mathcal{S}} \cap (\prod_{U^{ab} \in \Sigma(\mathcal{G})} \mathbb{I}(U^{ab})^\times) = \Phi$ and $\Theta'_{\Sigma(\mathcal{G})}(K_1(\mathbb{I}[[\mathcal{G}]])_{\mathcal{S}}) \subset \Phi_{\mathcal{S}}$.*

Proof. Let $C := [\text{Sel}_{E_\infty}^*(\text{Ad}^0(\rho))]$. Consider the following commutative diagram:

$$(83) \quad \begin{array}{ccccccc} K'_1(\mathbb{I}[[\mathcal{G}]]) & \longrightarrow & K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) & \xrightarrow{\tilde{\partial}_{\mathcal{G}}} & K_0(\mathbb{I}[[\mathcal{G}]], \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}) & \longrightarrow & 0 \\ \downarrow \Theta_{\Sigma(\mathcal{G})} & & \downarrow \Theta'_{\Sigma(\mathcal{G})} & & \downarrow \Theta_0 & & \\ \prod_{U^{ab} \in \Sigma(\mathcal{G})} K'_1(\mathbb{I}(U^{ab})) & \longrightarrow & \prod_{U^{ab} \in \Sigma(\mathcal{G})} K'_1(\mathbb{I}(U^{ab})_{\mathcal{S}}) & \xrightarrow{\partial_{\mathcal{G}}} & \prod_{U^{ab} \in \Sigma(\mathcal{G})} K_0(\mathbb{I}(U^{ab}), \mathbb{I}(U^{ab})_{\mathcal{S}}) & \longrightarrow & 0 \end{array}$$

Let g be any element in $K_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$ such that $\partial_{\mathcal{G}}(g) = -C$. Since the Main conjecture is valid for the extension $E^{U^{ab}}/E$, there exists $\xi_{U^{ab}}$ such that it is the pre-image of the class $-[\text{Sel}_{E^{U^{ab}}}^*(\text{Ad}^0(\rho))]$. On the other hand, the commutativity of the square on the left also implies that under the map $\Theta'_{\Sigma(\mathcal{G})}$ the element $(g_{U^{ab}})$ is mapped to $-[\text{Sel}_{E^{U^{ab}}}^*(\text{Ad}^0(\rho))]$. Therefore the element $(g_{U^{ab}}^{-1} \xi_{U^{ab}})$ comes from the group $\prod_{U^{ab} \in \Sigma(\mathcal{G})} K_1(\mathbb{I}(U^{ab})) = \prod_{U^{ab} \in \Sigma(\mathcal{G})} \mathbb{I}(U^{ab})^\times$. The second condition therefore implies that $(g_{U^{ab}}^{-1} \xi_{U^{ab}}) \in \Phi$.

By the isomorphism in the first condition, we find that there exists $u \in K'_1(\mathbb{I}[[\mathcal{G}]])$, such that $\Theta_{\Sigma(\mathcal{G})}(u) = (g_{U^{ab}}^{-1} \xi_{U^{ab}})$. Since the map $\Theta_{\Sigma(\mathcal{G})}$ is injective, the map $K'_1(\mathbb{I}[[\mathcal{G}]]) \rightarrow K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$ is injective.

Now, we define $\xi_{\mathcal{G}} := ug$, and we claim that this is the p -adic L-function defined over $\mathbb{I}[[\mathcal{G}]]$ that satisfies the interpolation formula. Clearly, $\tilde{\partial}_{\mathcal{G}}(\xi_{\mathcal{G}}) = \tilde{\partial}_{\mathcal{G}}(u) + \tilde{\partial}_{\mathcal{G}}(g) = \tilde{\partial}_{\mathcal{G}}(g) = -C$ as u comes from an element of $K'_1(\mathbb{I}[[\mathcal{G}]])$. For the interpolation formula, for any Artin representation ρ of \mathcal{G} , consider the isomorphism $\rho \cong \sum_{i \in I} m_i \cdot \text{Ind}_{U_i}^{\mathcal{G}} \text{Ind}_{U_i^{ab}}^{U_i} \rho_i$ of virtual representations given by the condition (*) above. Then, we have,

$$\phi_{\kappa}(\xi_{\mathcal{G}})(\rho) = \prod_{i \in I} \phi_{\kappa}(\xi_{\mathcal{G}})(\text{Ind}_{U_i}^{\mathcal{G}} \text{Ind}_{U_i^{ab}}^{U_i} \rho_i)^{m_i} = \prod_{i \in I} \phi_{\kappa}(\xi_{U_i^{ab}})(\rho_i)^{m_i}.$$

On the other hand, if the Main conjecture is true over the extension, then there exists $\xi \in K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}})$. Let $\Theta'_{\Sigma(\mathcal{G})}(\xi) = (\xi_{U^{ab}}) \in \prod_{U^{ab} \in \Sigma(\mathcal{G})} \mathbb{I}(U^{ab})^\times$. Note that the image $(\xi_{U^{ab}}) \in \Phi_{\mathcal{S}}$. By the interpolation formula, it is easy to see that the element $\xi_{U^{ab}}$ is the p -adic L-function over U^{ab} . Therefore $\xi_{U^{ab}} \in \Phi_{\mathcal{S}}$.

This finishes the proof of the proposition. \square

We now fix a lift $\tilde{\Gamma}$ of Γ in \mathcal{G} . Then we can identify \mathcal{G} with $H \rtimes \Gamma$. Fix $e \in \mathbb{N}$ such that $\tilde{\Gamma}^{p^e} \subset Z(\mathcal{G})$, and put $\tilde{\mathcal{G}} := \mathcal{G}/\Gamma^{p^e}$ and $\mathcal{R} := \mathbb{I}(\tilde{\Gamma}^{p^e})$. Then $\mathbb{I}[[\mathcal{G}]] \cong \mathcal{R}[\tilde{\mathcal{G}}]^\tau$, the twisted group ring with multiplication

$$(h\tilde{\gamma}^a)^\tau (h'\tilde{\gamma}^b)^\tau = \tilde{\gamma}^{p^e[\frac{a+b}{b}]} (h\tilde{\gamma}^a \cdot h'\tilde{\gamma}^b)^\tau,$$

where g^τ is the image of $g \in G$ in $\mathcal{R}[\tilde{\mathcal{G}}]^\tau$ ([Kak13, §5.1.1, §5.1.2]).

The Ore set \mathcal{S} that we have considered in the formulation of the Main Conjecture over $\mathbb{I}[[\mathcal{G}]]$ contains a multiplicative set which is crucial in setting up the strategy to prove the Conjecture.

Lemma 7.3. *Let $Z := Z(\mathcal{G})$. Consider the subset $T = \mathbb{I}(Z) \setminus p\mathbb{I}(Z)$. Then T is a multiplicatively closed left and right Ore set of $\mathbb{I}[[\mathcal{G}]]$. Further, the inclusion of rings $\mathbb{I}[[\mathcal{G}]]_T \hookrightarrow \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$ is an isomorphism.*

Proof. As Z is central in \mathcal{G} , it is easy to see that T is a multiplicatively Ore set. Further, T has no zero-divisors as it is contained in the domain $\mathbb{I}(Z)$. Therefore, the natural map $\mathbb{I}[[\mathcal{G}]]_T \rightarrow \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$ induced by the inclusion $T \hookrightarrow \mathcal{S}$ is an injective.

For surjectivity, consider the equality $\mathbb{I}[[\mathcal{G}]]_T = \mathbb{I}(Z)_T \otimes_{\mathbb{I}(Z)} \mathbb{I}[[\mathcal{G}]]$. We first observe that $Q(\mathbb{I}(Z)) \otimes_{\mathbb{I}(Z)} \mathbb{I}[[\mathcal{G}]] = Q(\mathbb{I}[[\mathcal{G}]])$. Indeed, it is easy to see that

$$Q(\mathbb{I}(Z)) \otimes_{\mathbb{I}(Z)} \mathbb{I}[[\mathcal{G}]] \hookrightarrow Q(\mathbb{I}[[\mathcal{G}]])$$

Further, the ring $\mathbb{I}[[\mathcal{G}]] = \mathbb{I}(Z)[\overline{\mathcal{G}}]$ is a module of finite rank over $\mathbb{I}(Z)$ and $Q(\mathbb{I}(Z))$ is a field, so the ring $Q(\mathbb{I}(Z)) \otimes_{\mathbb{I}(Z)} \mathbb{I}[[\mathcal{G}]]$ is Artinian. It follows that every regular element is a unit. The inclusion $\mathbb{I}[[\mathcal{G}]] \hookrightarrow Q(\mathbb{I}(Z)) \otimes_{\mathbb{I}(Z)} \mathbb{I}[[\mathcal{G}]]$, then implies that every regular element of $\mathbb{I}[[\mathcal{G}]]$ is invertible in $Q(\mathbb{I}(Z)) \otimes_{\mathbb{I}(Z)} \mathbb{I}[[\mathcal{G}]]$. It follows that the inclusion $Q(\mathbb{I}(Z)) \otimes_{\mathbb{I}(Z)} \mathbb{I}[[\mathcal{G}]] \hookrightarrow Q(\mathbb{I}[[\mathcal{G}]])$ is surjective.

Finally, if $x \in \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}} \subset Q(\mathbb{I}[[\mathcal{G}]])$, then $x = a/t$, for some $a \in \mathbb{I}[[\mathcal{G}]]$ and $t \in \mathbb{I}(Z)$ with $t \neq 0$. Here, if $t \in p^n \mathbb{I}(Z)$, then $tx = a \in p^n \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$. Since $a \in p^n \mathbb{I}[[\mathcal{G}]]$, it follows that $a \in p^n \mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}$. Cancelling the powers of p from a and t , the element $x = a'/t'$ with $t' \in T$. \square

Remark 7.4. The p -adic completion of $\mathbb{I}[[\mathcal{G}]]_T$ is denoted by $\widehat{\mathbb{I}[[\mathcal{G}]]}_T = \widehat{\mathbb{I}(Z)}_T [\overline{\mathcal{G}}]^\tau$. We also note that the localizations with respect to T and p are equal $\mathbb{I}(Z)_T = \mathbb{I}(Z)_{(p)}$.

Let P be a subgroup of $\overline{\mathcal{G}}$ and U_P be the inverse image of P in \mathcal{G} . Let

- (i) $N_{\overline{\mathcal{G}}}P :=$ the normalizer of P in \mathcal{G} ,
- (ii) $W_{\overline{\mathcal{G}}}(P) := N_{\overline{\mathcal{G}}}P/P$,
- (iii) $C(\overline{\mathcal{G}}) :=$ set of cyclic subgroups of $\overline{\mathcal{G}}$,
- (iv) for $P \in C(\overline{\mathcal{G}})$, the set $C_P(\overline{\mathcal{G}})$ denotes the set of cyclic subgroups P' of $\overline{\mathcal{G}}$ with $P'^p = P$ and $P' \neq P$.

If $P \in C(\overline{\mathcal{G}})$, then U_P is a rank one abelian subquotient of \mathcal{G} , and for every $P \in C(\overline{\mathcal{G}})$ set

$$(84) \quad \mathcal{T}_P := \left\{ \sum_{g \in W_{\overline{\mathcal{G}}}(P)} g^\tau x (g^\tau)^{-1} \mid x \in \mathcal{R}[P]^\tau \right\}.$$

In the same way, we define $\mathcal{T}_{P,\mathcal{S}}$ and $\widehat{\mathcal{T}}_P$.

Let $P \leq P' \leq \overline{\mathcal{G}}$. Then consider the homomorphism $\mathbb{I}[[\mathcal{G}]] \rightarrow \mathbb{I}[[\mathcal{G}]]$ given by $x \mapsto \sum_{g \in P'/P} \tilde{g}x\tilde{g}^{-1}$, where \tilde{g} is a lift of g . We define $\mathcal{T}_{P,P'}$ to be the image of this homomorphism. Similarly, we define $\mathcal{T}_{P,P',\mathcal{S}}$ and $\widehat{\mathcal{T}}_{P,P',\mathcal{S}}$, by considering the images of the same map on $\mathbb{I}(U_P)_{\mathcal{S}} \rightarrow \mathbb{I}(U_P)_{\mathcal{S}}$ and $\widehat{\mathbb{I}(U_P)}_{\mathcal{S}} \rightarrow \widehat{\mathbb{I}(U_P)}_{\mathcal{S}}$.

For two subgroups P, P' of $\overline{\mathcal{G}}$ with $[P', P'] \leq P \leq P'$ consider

$$(85) \quad \begin{aligned} \text{Tr}_P^{P'} : \mathbb{I}(U_P^{ab}) &\rightarrow \mathbb{I}(U_P/[U_{P'}, U_{P'}]), \quad (\text{the trace map}), \\ \text{Nr}_P^{P'} : \mathbb{I}(U_P^{ab})^\times &\rightarrow \mathbb{I}(U_P/[U_{P'}, U_{P'}])^\times, \quad (\text{the norm map}), \\ \Pi_P^{P'} : \mathbb{I}(U_P^{ab}) &\rightarrow \mathbb{I}(U_P/[U_{P'}, U_{P'}]), \quad (\text{the projection map}). \end{aligned}$$

We also have these maps in the localized case and the p -adic completion case. We continue to denote them by $\text{Tr}_P^{P'}$, $\text{Nr}_P^{P'}$ and $\Pi_P^{P'}$.

Recall the map $\Theta_{\Sigma(\mathcal{G})}$. For every subgroup P of $\overline{\mathcal{G}}$, let U_P denote the inverse image of P in \mathcal{G} . Then we have the following natural homomorphism

$$(86) \quad \Theta_P^{\mathcal{G}} : K_1'(\mathbb{I}[[\mathcal{G}]]) \rightarrow K_1'(\mathbb{I}(U_P)) \rightarrow K_1'(\mathbb{I}(U_P^{ab})) = \mathbb{I}(U_P^{ab})^\times.$$

Combining all these homomorphisms, we get the following homomorphism

$$(87) \quad \Theta^{\mathcal{G}} = (\Theta_P^{\mathcal{G}})_{P \leq \overline{\mathcal{G}}} : K_1'(\mathbb{I}[[\mathcal{G}]]) \rightarrow \prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})^\times.$$

Similarly, we also consider the following homomorphisms:

$$(88) \quad \Theta_{\mathcal{G}}^{\mathcal{G}} : K'_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{G}}) \longrightarrow \prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})_{\mathcal{G}}^{\times}$$

and

$$(89) \quad \widehat{\Theta}^{\mathcal{G}} : K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]}) \longrightarrow \prod_{P \leq \overline{\mathcal{G}}} \widehat{\mathbb{I}(U_P^{ab})}^{\times}.$$

For $P \in C(\overline{\mathcal{G}})$ with $P \neq (1)$, fix a homomorphism $\omega_P : P \longrightarrow \overline{\mathbb{Q}}_p^{\times}$ of order p , and also a homomorphism $\omega_1 := \omega_{\{1\}} : \widetilde{\Gamma}^{p^e} \longrightarrow \overline{\mathbb{Q}}_p^{\times}$ of order p . The homomorphism ω_P induce the following homomorphism which we again denote by the same symbol:

$$(90) \quad \omega_P : \mathbb{I}[\mu_p](U_P)^{\times} \longrightarrow \mathbb{I}[\mu_p](U_P)^{\times}, g \mapsto \omega_P(g)g.$$

For $P \leq \overline{\mathcal{G}}$, consider the homomorphism $\alpha_P : \mathbb{I}(U_P)_{\mathcal{G}}^{\times} \longrightarrow \mathbb{I}(U_P)_{\mathcal{G}}^{\times}$ defined by

$$(91) \quad \alpha_P(x) := \begin{cases} x^p & \text{if } P = \{1\} \\ x^p (\prod_{k=0}^{p-1} \omega_P^k(x))^{-1} & \text{if } P \neq \{1\} \text{ and cyclic} \\ x^p & \text{if } P \text{ is not cyclic.} \end{cases}$$

Note that, for all $P \leq \overline{\mathcal{G}}$, there is an action of \mathcal{G} and $\overline{\mathcal{G}}$ act on U_P^{ab} by conjugation since $\widetilde{\Gamma}^{p^e}$ is central. The following theorem is a generalization of results of Kakde, Kato, Burns, and Ritter and Weiss to $\mathbb{I}[[\mathcal{G}]]$ -modules.

Theorem 7.5. *Let \mathcal{G} be a rank one pro- p group. Then the set $\Sigma(\mathcal{G}) := \{U_P^{ab} : P \leq \overline{\mathcal{G}}\}$ satisfies the condition (*). Further, an element $(\Xi_{\mathcal{A}})_{\mathcal{A}} \in \prod_{\mathcal{A} \in \Sigma(\mathcal{G})} \mathbb{I}(\mathcal{A})^{\times}$ belongs to $\text{im}(\Theta_{\mathcal{G}})$ if and only if it satisfies all of the following three conditions.*

(i) *For all subgroups P, P' of $\overline{\mathcal{G}}$ with $[P', P'] \leq P \leq P'$, one has*

$$(92) \quad \text{Nr}_P^{P'}(\Xi_{U_{P'}^{ab}}) = \Pi_P^{P'}(\Xi_{U_{P'}^{ab}}).$$

(ii) *For all subgroups P of $\overline{\mathcal{G}}$ and all g in $\overline{\mathcal{G}}$ one has $\Xi_{gU_P^{ab}g^{-1}} = g\Xi_{U_P^{ab}}g^{-1}$.*

(iii) *For every $P \in C(\overline{\mathcal{G}})$ and $P \neq (1)$, we have*

$$\text{ver}_P^{P'}(\Xi_{U_{P'}^{ab}}) \equiv \Xi_{U_P^{ab}} \pmod{\mathcal{T}_{P,P'}} \text{ (resp. } \mathcal{T}_{P,P',\mathcal{G}} \text{ and } \widehat{\mathcal{T}}_{P,P'}).$$

(iv) *For all $P \in C(\overline{\mathcal{G}})$ one has $\alpha_P(\Xi_{U_P^{ab}}) \equiv \prod_{P' \in C_P(\overline{\mathcal{G}})} \alpha_{P'}(\Xi_{U_{P'}^{ab}}) \pmod{p\mathcal{T}_P}$.*

We give a proof of this theorem referring to [Kak13] for many of the details which remain true in our set-up.

Definition 7.6. Let $\Phi_{\mathcal{H}}^{\overline{\mathcal{G}}}$ (resp. $\Phi_{\mathcal{H},\mathcal{G}}^{\overline{\mathcal{G}}}$ and $\widehat{\Phi}_{\mathcal{H}}^{\overline{\mathcal{G}}}$) denote the subgroup of $\prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})^{\times}$ (resp. $\prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})_{\mathcal{G}}^{\times}$ and $\prod_{P \leq \overline{\mathcal{G}}} \widehat{\mathbb{I}(U_P^{ab})}_{\mathcal{G}}^{\times}$) consisting of tuples $(\Xi_{U_P^{ab}})$ satisfying the conditions of the above theorem:

(C1) For all subgroups P, P' of $\overline{\mathcal{G}}$ with $[P', P'] \leq P \leq P'$, one has

$$\text{Nr}_P^{P'}(\Xi_{U_{P'}^{ab}}) = \Pi_P^{P'}(\Xi_{U_{P'}^{ab}}).$$

(C2) For all subgroups P of $\overline{\mathcal{G}}$ and all g in $\overline{\mathcal{G}}$ one has $\Xi_{gU_P^{ab}g^{-1}} = g\Xi_{U_P^{ab}}g^{-1}$.

(C3) For every $P \in \overline{\mathcal{G}}$ and $P \neq (1)$, we have

$$\text{ver}_P^{P'}(\Xi_{U_{P'}^{ab}}) \equiv \Xi_{U_P^{ab}} \pmod{\mathcal{T}_{P,P'}} \text{ (resp. } \mathcal{T}_{P,P',\mathcal{G}} \text{ and } \widehat{\mathcal{T}}_{P,P'}).$$

(C4) For all $P \in C(\overline{\mathcal{G}})$ one has $\alpha_P(\Xi_{U_P^{ab}}) \equiv \prod_{P' \in C_P(\overline{\mathcal{G}})} \alpha_{P'}(\Xi_{U_{P'}^{ab}}) \pmod{p\mathcal{T}_P}$ (resp. $p\mathcal{T}_{P,\mathcal{J}}$ and $p\widehat{\mathcal{T}}_P$).

As in [Kak13, Kat06, Bur15, RW08], the theorem follows from an explicit description of the image of the groups $K_1(\mathbb{I}[[\mathcal{G}]])$ and $K_1(\mathbb{I}[[\mathcal{G}]]_{\mathcal{J}^*})$. We follow the same steps as in [Kak13] to prove this theorem. In fact, the theorem is a combination of the following two theorems which are generalizations of [Kak13, Theorem 52 and 53]. We will give the main results leading to a proof of these theorems.

For any $P \leq \overline{\mathcal{G}}$, consider the map

$$t_P^{\overline{\mathcal{G}}} : \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau \longrightarrow \mathcal{R}[P^{ab}]^\tau$$

defined by

$$t_P^{\overline{\mathcal{G}}}(\bar{g}) = \sum_{x \in C(\overline{\mathcal{G}}, P)} \{(\bar{x}^{-1})(\bar{g})(\bar{x}) \mid x^{-1}gx \in P\},$$

where $C(\overline{\mathcal{G}}, P)$ is the set of left coset representatives of P in $\overline{\mathcal{G}}$. This is a well-defined \mathcal{R} -linear map, independent of the choice of $C(\overline{\mathcal{G}}, P)$. For any $P \in C(\overline{\mathcal{G}})$, we define

$$\eta_P : \mathcal{R}[P]^\tau \longrightarrow \mathcal{R}[P]^\tau,$$

by \mathcal{R} -linearly extending the map,

$$\eta_P(h) = \begin{cases} h & \text{if } h \text{ is a generator of } P \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\eta_P(x) = x - \frac{1}{p} \sum_{k=0}^{p-1} \omega_P^k(x)$.

Now, define the homomorphism $\beta_P^{\overline{\mathcal{G}}} : \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau \longrightarrow \mathcal{R}[P^{ab}]^\tau$ by

$$\beta_P^{\overline{\mathcal{G}}} = \begin{cases} \eta_P \circ t_P^{\overline{\mathcal{G}}} & \text{if } P \in C(\overline{\mathcal{G}}) \\ t_P^{\overline{\mathcal{G}}} & \text{if } P \leq \overline{\mathcal{G}} \text{ is not cyclic} \end{cases}$$

and $\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ is defined by

$$\beta_{\mathcal{R}}^{\overline{\mathcal{G}}} = (\beta_P^{\overline{\mathcal{G}}})_{P \leq \overline{\mathcal{G}}} : \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau \longrightarrow \prod_{P \leq \overline{\mathcal{G}}} \mathcal{R}[P^{ab}]^\tau.$$

Definition 7.7. Let $\Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}$ (resp. $\Psi_{\mathcal{R},\mathcal{J}}^{\overline{\mathcal{G}}}$) be the subgroup of $\prod_{P \leq \overline{\mathcal{G}}} (\mathcal{R}[P^{ab}]^\tau)^\times$ (resp. $\prod_{P \leq \overline{\mathcal{G}}} (\mathcal{R}[P^{ab}]^\tau)_{\mathcal{J}}^\times$) consisting of all tuples (a_P) with the following properties:

(A1) Let $P \leq P' \leq \overline{\mathcal{G}}$ such that $[P', P'] \leq P$ and the following conditions hold:

- (a) if P is a non-trivial cyclic group then $[P', P'] \neq P$;
- (b) if P is not cyclic, then $\text{tr}_P^{P'}(a_{P'}) = \pi_{P'}^{P'}(a_P)$;
- (c) if P is cyclic but P' is not cyclic then $\eta_P(\text{tr}_P^{P'}(a_{P'})) = \pi_P^{P'}(a_P)$;
- (d) if P' is cyclic, then $\text{tr}_P^{P'}(a_{P'}) = 0$.

(A2) $(a_P)_{P \in C(\overline{\mathcal{G}})}$ is invariant under conjugation action by every $g \in \overline{\mathcal{G}}$.

(A3) For all $P \in C(\overline{\mathcal{G}})$, $a_P \in \mathcal{T}_P$.

Then we have the following theorem as a generalization of [Kak13, Theorem 58].

Theorem 7.8. *The homomorphism $\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ induces an isomorphism between $\mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$ and $\Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}$.*

The first step of the proof is to show that the image of $\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ is contained in $\Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}$. This proof is the same as the proof of [Kak13, Lemma 60]. The next step is to consider the following map and get left inverse of $\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$.

$$\begin{aligned} \delta_P : \mathcal{R}[P^{ab}]^\tau &\longrightarrow \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau \begin{bmatrix} 1 \\ p \end{bmatrix} \\ x &\mapsto \begin{cases} \frac{1}{[\overline{\mathcal{G}}:P]}[x], & \text{if } P \text{ is cyclic} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Combining all these maps, we get the following map:

$$\begin{aligned} \delta : \prod_{P \leq \overline{\mathcal{G}}} \mathcal{R}[P^{ab}]^\tau &\longrightarrow \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau \begin{bmatrix} 1 \\ p \end{bmatrix}, \\ \delta &= \sum_{P \leq \overline{\mathcal{G}}} \delta_P. \end{aligned}$$

Lemma 7.9. *The composite map $\delta \circ \beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ is identity on $\mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$. In particular, the map $\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ is injective.*

Proof. Let $g \in \overline{\mathcal{G}}$ and $P = \langle g \rangle$, and consider the collection C of all the conjugates of P in $\overline{\mathcal{G}}$. Then,

$$\begin{aligned} \delta(\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}([g])) &= \sum_{P' \in C} \delta_{P'}(\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}([g])) \\ &= \sum_{P' \in C} \frac{1}{[\overline{\mathcal{G}}:P]} [\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}([g])] \\ &= \frac{1}{[\overline{\mathcal{G}}:P]} \sum_{P' \in C} [N_{\overline{\mathcal{G}}}P' : P'] [g] \\ &= \frac{1}{[\overline{\mathcal{G}}:N_{\overline{\mathcal{G}}}P]} \sum_{P' \in C} [g] \\ &= [g]. \end{aligned}$$

□

Next, in a similar way as in [Kak13, Lemma 63], we get the following lemma:

Lemma 7.10. *The map $\delta|_{\Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}}$ is injective and its image lies in $\mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$.*

Finally, we can show Theorem 7.8.

Proof. Since $\delta \circ \beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ is identity on $\mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$ and $\delta|_{\Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}}$ is injective, $\delta \circ \beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ is also identity on $\Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}$. Indeed, if $(a_P) \in \Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}$, then $\delta(\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}(\delta((a_P)))) = \delta((a_P))$. As the image of $\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ is contained in $\Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}$ and δ is injective on $\Psi_{\mathcal{R}}^{\overline{\mathcal{G}}}$, we have $\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}(\delta((a_P))) = (a_P)$. Therefore $\beta_{\mathcal{R}}^{\overline{\mathcal{G}}}$ is surjective.

By Artin's induction theorem, a linear representation of a finite group is a \mathbb{Q} -linear combination of representations induced from cyclic subgroups [Ser77, Theorem 17]. The injectivity follows using this result. □

Proposition 7.11. *Let K be the quotient field of \mathcal{O} . Then the map $\text{id}_K \otimes \beta_{\mathcal{R}}^{\overline{\mathcal{G}}} : K \otimes \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau \longrightarrow \prod_{P \leq \overline{\mathcal{G}}} K \otimes \mathcal{R}[P^{ab}]^\tau$ is injective, and its image consists of all tuples (a_P) satisfying the following:*

- (i) Let $P \leq P' \leq \mathcal{G}$ such that $[P', P'] \leq P$ and the following conditions hold:
 - (a) if P is a non-trivial cyclic group then $[P', P'] \neq P$;

- (b) if P is not cyclic, then $\text{tr}_P^{P'}(a_{P'}) = \pi_P^{P'}(a_P)$;
- (c) if P is cyclic but P' is not cyclic then $\eta_P(\text{tr}_P^{P'}(a_{P'})) = \pi_P^{P'}(a_P)$;
- (d) if P' is cyclic, then $\text{tr}_P^{P'}(a_{P'}) = 0$.

(ii) $(a_P)_{P \in C(\overline{\mathcal{G}})}$ is invariant under conjugation action by every $g \in \overline{\mathcal{G}}$.

Hence, if $\text{id}_K \otimes \beta_{\mathcal{R}}^{\overline{\mathcal{G}}}(a) = (a_P)$ with $a_P \in \mathcal{T}_P, \forall P \in C(\overline{\mathcal{G}})$, then $a \in \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$ and $a_P \in \mathcal{R}[P^{ab}]^\tau, \forall P \leq \overline{\mathcal{G}}$.

Proof. By Lemma 7.10 above, the injectivity is clear. The statement about the image also follows from this Lemma. Clearly, if $\text{id}_K \otimes \beta_{\mathcal{R}}^{\overline{\mathcal{G}}}(a) = (a_P)$, with $a_P \in \mathcal{T}_P, \forall P \in C(\overline{\mathcal{G}})$, then as the map δ is determined by the a_P 's for cyclic P , it follows that the inverse image a lies in $\mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$ and $a_P \in \mathcal{R}[P^{ab}]^\tau, \forall P \leq \overline{\mathcal{G}}$. \square

7.2 The Logarithm map over $\mathbb{I}(\tilde{\Gamma}^{p^e})$

Recall that $\mathcal{R} := \mathbb{I}[\tilde{\Gamma}^{p^e}]$. Note that \mathcal{R} is a local ring. Our aim in this section is to construct a logarithm map on $K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau)$. This is done by generalizing the logarithm map which was considered by Ritter and Weiss and then later by Kakde. Their constructions were inspired by the logarithm map introduced by Oliver.

We assume that \mathcal{G} is a p -adic Lie group of rank 1. For any subgroup P of $\overline{\mathcal{G}}$, we set

$$(93) \quad \mathcal{R}_P := \mathbb{I}[U_P] = \mathbb{I}[\tilde{\Gamma}^{p^e}][P]^\tau = \mathcal{R}[P]^\tau.$$

Consider the natural \mathcal{R} -linear map

$$\kappa_P : \mathcal{R}_P \longrightarrow \mathcal{R}[\text{Conj}(P)]^\tau, \kappa_P(g^\tau) = [g^\tau],$$

for all $g \in P$, where $[g^\tau]$ denotes the conjugacy class in P .

For any ring A , let $J(A)$ denote the Jacobson radical. Since $\mathcal{G} = H \rtimes \Gamma$, the kernel of the composite homomorphism $U_P \hookrightarrow \mathcal{G} \twoheadrightarrow \Gamma$ is $H \cap U_P$. Let $I_{H \cap U_P}$ be the augmentation ideal of $\mathcal{R}_{H \cap U_P}$. Then for the ring \mathcal{R}_P , the Jacobson radical $J(\mathcal{R}_P)$ is generated by $\mathfrak{m}_{\mathbb{I}}$ and $I_{H \cap U_P}$, where $\mathfrak{m}_{\mathbb{I}}$ denotes the maximal ideal $\langle p, X_1, \dots, X_r \rangle$ of \mathbb{I} .

In the proof of the following propositions, the following short exact sequence which is obtained by using $\mathcal{R}_P/J(\mathcal{R}_P) \cong \mathbb{F}_p$ play a crucial role:

$$(94) \quad 1 \longrightarrow K_1(\mathcal{R}_P, J(\mathcal{R}_P)) \longrightarrow K_1(\mathcal{R}_P) \longrightarrow K_1(\mathbb{F}_p).$$

It is well known that $K_1(\mathbb{F}_p) \cong \mathbb{F}_p^\times$.

Proposition 7.12. *Let $P \leq \overline{\mathcal{G}}$. Then, for $x \in J(\mathcal{R}_P)$, the logarithm defined by*

$$(95) \quad \text{Log}(1+x) := \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}.$$

is well-defined, and it induces a homomorphism

$$\log_P : K_1(\mathcal{R}_P) \longrightarrow \mathcal{R}[\text{Conj}(P)]^\tau \left[\frac{1}{p} \right].$$

Moreover, this map is natural with respect to ring homomorphisms induced by group homomorphisms.

Proof. First, we show that the map is well-defined by showing that the power series $\text{Log}(1+x)$ converges in $\mathcal{R}[P]^\tau \left[\frac{1}{p} \right]$.

Since $H \cap U_P$ is a finite p -group, with say, p^r elements, we have $(g-1)^{p^r} \in p\mathbb{I}[H \cap U_P]$, for any $g \in H \cap U_P$. Therefore, for any $x \in J(\mathcal{R}_P) = \langle \mathfrak{m}_{\mathbb{I}}, I_{H \cap U_P} \rangle$, we have $x^{p^r} \in \langle p, \mathfrak{m}_{\mathbb{I}} \rangle \mathcal{R}_P$.

Hence $x^n \in \langle p, \mathfrak{m}_1 \rangle^m \mathcal{R}_P$ for large enough n, m . This implies that x^i/i converges to 0 as i tends to infinity. Hence the series $\text{Log}(1+x)$ converges in $\mathcal{R}[P]^\tau[\frac{1}{p}]$.

We now use arguments of Oliver to construct the map \log_P . Indeed, the proof of Oliver for [Oli88, Lemma 2.7], shows that for any $x, y \in J(\mathcal{R}_P)$, we have

$$\text{Log}((1+x)(1+y)) \equiv \text{Log}(1+x) + \text{Log}(1+y) \pmod{[\mathcal{R}_P[\frac{1}{p}], J(\mathcal{R}_P)[\frac{1}{p}]]}.$$

Then by the proof of [Oli88, Theorem 2.8], $\text{Log}(1+x)$ induces a well-defined homomorphism

$$\log'_P : K_1(\mathcal{R}_P, J(\mathcal{R}_P)) \longrightarrow (J(\mathcal{R}_P)/[\mathcal{R}_P, J(\mathcal{R}_P)])[\frac{1}{p}].$$

Since $\mathcal{R}_P/J(\mathcal{R}_P) \cong \mathbb{F}_q$, we have the following exact sequence

$$(96) \quad 1 \longrightarrow K_1(\mathcal{R}_P, J(\mathcal{R}_P)) \longrightarrow K_1(\mathcal{R}_P) \longrightarrow K_1(\mathbb{F}_q).$$

Here $K_1(\mathbb{F}_q) \cong \mathbb{F}_q^\times$ and $(J(\mathcal{R}_P)/[\mathcal{R}_P, J(\mathcal{R}_P)])[\frac{1}{p}]$ is torsion-free. Hence the map \log'_P can be extended uniquely to $K_1(\mathcal{R}_P)$, which we call \log_P . \square

Remark 7.13. The proof of [Oli88, Theorem 2.8] can also be generalized to show that we have the following homomorphism:

$$(97) \quad \log_P^I : K_1(\mathcal{R}_P, I) \longrightarrow (I/[\mathcal{R}_P, I]) \left[\frac{1}{p} \right],$$

for any ideal $I \subset J(\mathcal{R}_P)$.

Lemma 7.14. *Let U_P be abelian and let I be any ideal of $\mathcal{R}[P]^\tau$ such that $I \subset p\mathcal{R}[P]^\tau$. Then \log_P is a well-defined map from $1+I$ to I , which is an isomorphism.*

Proof. We first show that the map is well-defined. Since $I \subseteq p\mathcal{R}[P]^\tau$, we have $I^p \subseteq pI$. It follows that $I^{p^r} \subseteq p^r I$ for all $r \in \mathbb{N}$. This further implies that $I^n \subseteq nI$, for all $n \in \mathbb{N}$. This is easy to see if $p \nmid n$, as n is then a unit and $I^n \subseteq I = nI$. Next, if $\text{ord}_p(n) = e$, then $I^{p^e} \subseteq p^e I$. As n/p^e is a unit, raising to n/p^e -th power, we have $I^n \subseteq nI$. Now, let $x \in I$, then $x^n \in I^n \subseteq nI$. Therefore $x^n/n \in I$. Noting that the series $\sum_{n \geq 0} (-1)^{n+1} \frac{x^n}{n}$ converges, it converges in I .

Conversely, we show that for each $x \in I$, the series $\exp_P(x) = \sum_{n \geq 0} \frac{x^n}{n!}$ is convergent in $1+I$. Since $I \subset p\mathcal{R}[P]^\tau \cong \mathbb{I}(U_P)$, the Jacobson radical $J(\mathcal{R}[P]^\tau)$ contains I . Further, $I^p \subseteq pI \cdot J(\mathcal{R}[P]^\tau)$. Suppose $I^{p^k} \subseteq p^k I \cdot J(\mathcal{R}[P]^\tau)^k$, then raising both sides to a p -th power, we get

$$I^{p^{k+1}} \subseteq (p^k!)^p I^p \cdot J(\mathcal{R}[P]^\tau)^{kp}.$$

Clearly $(p^k!)^p I^p \cdot J(\mathcal{R}[P]^\tau)^{kp} \subseteq (p^{k+1}!) I \cdot J(\mathcal{R}[P]^\tau)^{k+1}$. Therefore, by induction,

$$I^{p^n} \subseteq p^n! I \cdot J(\mathcal{R}[P]^\tau)^n, \text{ for all } n \in \mathbb{N}.$$

The lemma follows by observing that \log_P and \exp_P are inverses of each other. \square

For each pair of subgroups P and P' of $\bar{\mathcal{G}}$ with $P \leq P'$, consider the natural *restriction map* on K -groups

$$(98) \quad \Theta_P^{P'} : K_1(\mathbb{I}(U_{P'})) = K_1(\mathcal{R}_{P'}) \longrightarrow K_1(\mathcal{R}_P) = K_1(\mathbb{I}(U_P)).$$

Moreover, we define an \mathbb{I} -linear map

$$(99) \quad \text{Res}_P^{P'} : \mathcal{R}[\text{Conj}(P')]^\tau \longrightarrow \mathcal{R}[\text{Conj}(P)]^\tau$$

given by

$$(100) \quad \text{Res}_P^{P'}(\kappa_{P'}(g^\tau)) := \sum_x \kappa_P((x^\tau)^{-1}(gx)^\tau)$$

where x runs over all elements in a given set of left coset representatives of P in P' with $xgx^{-1} \in P$.

Lemma 7.15. *For each subgroup P of $\overline{\mathcal{G}}$, we have the following commutative diagram.*

$$\begin{array}{ccc} K_1(\mathbb{I}[[\mathcal{G}]]) & \xrightarrow{\log_{\overline{\mathcal{G}}}} & \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau[\frac{1}{p}] \\ \downarrow \Theta_P^{\overline{\mathcal{G}}} & & \downarrow \text{Res}_P^{\overline{\mathcal{G}}} \\ K_1(\mathbb{I}(U_P)) & \xrightarrow{\log_P} & \mathcal{R}[\text{Conj}(P)]^\tau[\frac{1}{p}]. \end{array}$$

Proof. For any $\xi \in \mathbb{I}[[\mathcal{G}]]$, it follows from a similar argument as used by Oliver and Taylor to prove [OT88, Theorem 1.4], that

$$(101) \quad \log_P(\Theta_P^{\overline{\mathcal{G}}}(1 + p\xi)) = \text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1 + p\xi)).$$

Recall that $\mathbb{I} \cong \mathcal{O}[[X_1, \dots, X_r]]$ and consider now $x \in J(\mathcal{R}[\overline{\mathcal{G}}]^\tau)$. We can see that $J(\mathcal{R}[\overline{\mathcal{G}}]^\tau)$ is generated by $\mathfrak{m}_{\mathbb{I}}$ and the augmentation ideal I_H of $\Lambda(H)[[X_1, \dots, X_r]]$. Therefore, $I_H^n \subseteq p\Lambda(H) = p\mathbb{Z}_p[H]$, for n sufficiently large.

Now, for n sufficiently large, we have $(1+x)^{p^n} = 1 + p\xi'' + x^{p^n}$. Further, $x = a + (\sum_j b_j X_j) + c$, where $a \in p\Lambda(H), b \in \Lambda(H), c \in I_H$. Hence, $x^{p^n} = a^{p^n} + \sum_j b_j^{p^n} X_j^{p^n} + c^{p^n} + pd$, where $d \in \Lambda(H)[[X_1, \dots, X_r]]$. As $c \in I_H$, we can see that $x^{p^n} = a^{p^n} + \sum_j b_j^{p^n} X_j^{p^n} + pd'$, for some $d' \in \Lambda(H)[[X_1, \dots, X_r]]$. This means that we can put $(1+x)^{p^n}$ in the following form

$$(1+x)^{p^n} = 1 + p\xi'' + x^{p^n} = 1 + p\xi + \sum_j X_j^{p^m} \xi'_j = (1 + \sum_j X_j^{p^m} \xi'_j)[1 + p\xi(1 + \sum_j X_j^{p^m} \xi'_j)^{-1}],$$

for some $\xi, \xi', \xi'' \in \mathcal{R}[\overline{\mathcal{G}}]^\tau$. Here $\sum_j X_j^{p^m} \xi'_j \in J(\mathcal{R}[\overline{\mathcal{G}}]^\tau)$ and $(1 + \sum_j X_j^{p^m} \xi'_j)^{-1} \in \mathcal{R}[\overline{\mathcal{G}}]^\tau$. It follows from (101) that

$$\begin{aligned} & p^m \log_P(\Theta_P^{\overline{\mathcal{G}}}(1+x)) \\ &= \log_P(\Theta_P^{\overline{\mathcal{G}}}(1+x)^{p^m}) \\ &= \log_P(\Theta_P^{\overline{\mathcal{G}}}(1 + p\xi(1 + \sum_j X_j^{p^m} \xi'_j)^{-1})) + \log_P(\Theta_P^{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j)) \\ &= \text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1 + p\xi(1 + \sum_j X_j^{p^m} \xi'_j)^{-1})) + \log_P(\Theta_P^{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j)) \\ &= \text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1+x)^{p^m}) + \log_P(\Theta_P^{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j)) - \text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j)) \\ &= p^m \text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1+x)) + \log_P(\Theta_P^{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j)) - \text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j)). \end{aligned}$$

Here $\log_{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j) \in (X_1^{p^m}, \dots, X_r^{p^m})\mathcal{R}[\text{Conj}(P)]^\tau[\frac{1}{p}]$. Moreover, it is easy to see that

$$\Theta_P^{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j) \equiv 1 + \sum_j X_j^{p^m} \text{Res}_P^{\overline{\mathcal{G}}} \xi'_j \pmod{K_1(\mathcal{R}[P]^\tau, (X_1^{p^m}, \dots, X_r^{p^m})\mathcal{R}[P]^\tau)}.$$

Therefore, $\log_P(\Theta_P^{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j))$ and $\text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1 + \sum_j X_j^{p^m} \xi'_j))$ both belong to $(X_1^{p^m}, \dots, X_r^{p^m})\mathcal{R}[P]^\tau[\frac{1}{p}]$. Using this in the above equality, we get

$$\log_P(\Theta_P^{\overline{\mathcal{G}}}(1+x)) \equiv \text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1+x)) \pmod{(X_1^{p^m}, \dots, X_r^{p^m})\mathcal{R}[\text{Conj}(P)]^\tau[\frac{1}{p}]}.$$

Since this is true for all $m \in \mathbb{N}, m \gg 0$, therefore

$$\log_P(\Theta_P^{\overline{\mathcal{G}}}(1+x)) = \text{Res}_P^{\overline{\mathcal{G}}}(\log_{\overline{\mathcal{G}}}(1+x)).$$

Hence the diagram in the lemma commutes for the subgroup $K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau, J(\mathcal{R}[\overline{\mathcal{G}}]^\tau))$ of $K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau)$. Now note that the index of the subgroup $K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau, J(\mathcal{R}[\overline{\mathcal{G}}]^\tau))$ of $K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau)$ is finite, and the group $\mathcal{R}[\text{Conj}(P)]^\tau[\frac{1}{p}]$ is torsion-free. Therefore the commutativity extends to the diagram in the assertion of the lemma. \square

7.3 The Integral logarithm over $\mathbb{I}[[\mathcal{G}]]$

We now make the following assumption

(Ur) $\mathbb{I} = \mathcal{O}[[X_1, \dots, X_r]]$ with \mathcal{O} unramified over \mathbb{Z}_p .

Then on $\mathcal{R} = \mathbb{I}(\tilde{\Gamma}^{p^e})$, consider the map $\varphi : \mathcal{R} \rightarrow \mathcal{R}$ such that its restriction to \mathcal{O} is the Frobenius and maps X_j^n to X_j^{pn} for all $j = 1, \dots, r$ and the p -th power map on $\tilde{\Gamma}^{p^e}$. Further, we extend φ to a map

$$\varphi_{\text{conj}} : \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau \rightarrow \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau,$$

by

$$\kappa(g^\tau) \mapsto \kappa((g^\tau)^p).$$

Definition 7.16. The map $\mathfrak{L}_{\mathcal{G}} : K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau) \rightarrow \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau[\frac{1}{p}]$ defined by

$$\mathfrak{L}_{\mathcal{G}} := \log_{\mathcal{G}} - p^{-1}\varphi_{\text{conj}} \circ \log_{\mathcal{G}}$$

is called the p -adic logarithm map over $\mathbb{I}[[\mathcal{G}]]$.

Proposition 7.17. For the map $\mathfrak{L}_{\mathcal{G}} : K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau) \rightarrow \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau[\frac{1}{p}]$, we have $\text{Im}(\mathfrak{L}_{\mathcal{G}}) \subseteq \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$.

Proof. As before, by the exact sequence in (94), the index of $K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau, J(\mathcal{R}[\overline{\mathcal{G}}]^\tau))$ in $K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau)$ is finite and prime to p . Therefore, it is enough to prove that $\mathfrak{L}_{\mathcal{G}}(K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau, J(\mathcal{R}[\overline{\mathcal{G}}]^\tau))) \subseteq \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$.

Let $y \in K_1(\mathcal{R}[\overline{\mathcal{G}}]^\tau, J(\mathcal{R}[\overline{\mathcal{G}}]^\tau))$. Then, by [Vas69], there exists x such that $y = [1 - x]$ for some $x \in J(\mathcal{R}[\overline{\mathcal{G}}]^\tau)$. Then, we have,

$$\begin{aligned} \mathfrak{L}_{\mathcal{G}}(y) &= -\sum_{i \geq 1} \frac{x^i}{i} + \sum_{i \geq 1} \frac{\varphi_{\text{conj}}(x^i)}{pi} \\ &= -\sum_{i \geq 1, p \nmid i} \frac{x^i}{i} - \sum_{j \geq 1} \frac{x^{pj} - \varphi_{\text{conj}}(x^j)}{pj}. \end{aligned}$$

Therefore, it is enough to prove that the sum $\sum_{j \geq 1} \frac{x^{pj} - \varphi_{\text{conj}}(x^j)}{pj} \in \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau$. This follows from the same argument as in the proof of [Oli88, Theorem 6.2]. \square

We now extend the theorem of Oliver [Oli88, Theorem 6.4, 6.6] to \mathcal{R}_G , for finite groups G of prime order. For this, we recall the following exact sequence from [Oli88, Lemma 6.3(ii)]:

$$(102) \quad 0 \hookrightarrow \mathbb{F}_p \rightarrow \mathcal{R}/\mathfrak{m}_{\mathcal{R}} \xrightarrow{1-\varphi} \mathcal{R}/\mathfrak{m}_{\mathcal{R}} \xrightarrow{\text{Tr}} \mathbb{F}_p \rightarrow 0$$

where \mathbb{F}_p is the finite field of order p , φ is the Frobenius and Tr denotes the trace map. Here we note that \mathcal{R} is a local field with maximal ideal $\mathfrak{m}_{\mathcal{R}}$.

Proposition 7.18. Let G be a finite p -group and z be an element of order p in the center $Z(G)$. Then, we have the following exact sequence:

$$(103) \quad 1 \rightarrow \langle z \rangle \rightarrow K_1(\mathcal{R}_G, (1-z)\mathcal{R}_G) \xrightarrow{\log_{\mathcal{G}}} H_0(G, (1-z)\mathcal{R}_G) \xrightarrow{\omega} \mathbb{F}_p \rightarrow 0$$

Proof. The proof is the same as in the proof of [Oli88, Theorem 6.4]. We set $I = (1 - z)\mathcal{R}_G$ and J the Jacobson radical of \mathcal{R}_G . As $(1 - z)^p \in pI$, the p -adic logarithm induces a homomorphism \log_G^I and an isomorphism \log_G^{IJ} . These maps fit in the following commutative diagram:

$$\begin{array}{ccccccc} K_1(\mathcal{R}_G, (1 - z)J) & \longrightarrow & K_1(\mathcal{R}_G, I) & \longrightarrow & K_1\left(\frac{\mathcal{R}_G}{(1 - z)J}, \frac{I}{(1 - z)J}\right) & \longrightarrow & 1 \\ \log_G^{IJ} \downarrow \cong & & \log_G^I \downarrow & & \log_0 \downarrow & & \\ 0 \longrightarrow & H_0(G, (1 - z)J) & \longrightarrow & H_0(G, I) & \longrightarrow & H_0\left(G, \frac{I}{(1 - z)J}\right) & \longrightarrow 0 \end{array}$$

By a result of Bass (see [Oli88, Theorem 1.15]), we have the following identification

$$K_1\left(\frac{\mathcal{R}_G}{(1 - z)J}, \frac{I}{(1 - z)J}\right) \xrightarrow[\cong]{\alpha} \mathcal{R}_G/J \cong \mathcal{R}/\mathfrak{m}_{\mathcal{R}},$$

where the map α is given by $\alpha(1 + (1 - z)\xi) = \xi$ for $\xi \in \mathcal{R}_G/J$. Further, $H_0(G, I/(1 - z)J) \cong \mathcal{R}/\mathfrak{m}_{\mathcal{R}}$. Together with the exact sequence in (102), we can see that the map \log_0 fits in the following exact sequence:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathcal{R}_G/J \xrightarrow{\cong} K_1\left(\frac{\mathcal{R}_G}{(1 - z)J}, \frac{I}{(1 - z)J}\right) \xrightarrow{\log_0} H_0\left(G, \frac{I}{(1 - z)J}\right) \xrightarrow{\cong} \mathcal{R}/\mathfrak{m}_{\mathcal{R}} \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

as a result, we have the following commutative diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \mathbb{F}_p & & \\ & & & & \downarrow & & \\ K_1(\mathcal{R}_G, I) & \xrightarrow{\alpha'} & \mathcal{R}_G/J \cong \mathcal{R}/\mathfrak{m}_{\mathcal{R}} & \longrightarrow & 1 & & \\ \log_G^I \downarrow & & \log_0 \downarrow \cong & & & & \\ H_0(G, I) & \xrightarrow{\alpha''} & \mathcal{R}/\mathfrak{m}_{\mathcal{R}} & \longrightarrow & 0 & & \\ & \searrow \omega & \downarrow \text{Tr} & & & & \\ & & \mathbb{F}_p & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

where $\alpha'(1 + (1 - z)\sum r_i g_i) = \sum \bar{r}_i$ and $\alpha''((1 - z)\sum r_i g_i) = \sum \bar{r}_i$. Indeed, the right column is exact, and as

$$\alpha''(\log_G^I(1 + (1 - z)rg)) = \alpha''((1 - z)(rg - r^p g^p)) = r - \varphi(r) \in \mathcal{R}/\mathfrak{m}_{\mathcal{R}},$$

the square is also commutative. Therefore the maps \log_G^I and $1 - \varphi$ have isomorphic kernel and cokernel. Noting that $\omega = \text{Tr} \circ \alpha''$ and α' maps $\langle z \rangle$ isomorphically onto $\mathbb{F}_p = \ker(1 - \varphi)$, the exactness of the sequence in the Proposition follows. \square

We now prove the key result regarding the integral logarithm. This is a generalization of the following exact sequence, ([Kak13, Def 70]). Since $\mathbb{I} = \mathcal{O}[[X_1, \dots, X_r]]$, the ring $\mathcal{R} = \mathbb{I}[[\tilde{\Gamma}^{p^e}]]$ is isomorphic to an Iwasawa algebra with $r + 1$ variables. Let $\mathbb{W} \cong (1 + p\mathbb{Z}_p)^{r+1}$, then $\mathcal{R} \cong \mathcal{O}[[\mathbb{W}]]$.

We assume that the extension \mathcal{O}/\mathbb{Z}_p is unramified. Recall that the quotient field of \mathcal{O} is denoted by K . Then the following sequence is an exact sequence of groups:

$$1 \longrightarrow \mu_K \times \mathbb{W} \longrightarrow K_1(\mathcal{R}) \xrightarrow{\mathfrak{L}} \mathcal{R} \xrightarrow{\omega} \mathbb{W} \longrightarrow 1.$$

Proposition 7.19. *Let G be a finite p group. Let $\mathbb{W} \cong (1 + p\mathbb{Z}_p)^{r+1}$ and $\mathbb{I} = \mathcal{O}[[X_1, \dots, X_r]]$, such that the extension \mathcal{O}/\mathbb{Z}_p is unramified. Define the map*

$$\tilde{\omega} : \mathcal{R}[G] \longrightarrow \mathbb{W} \times G^{ab}$$

by $\tilde{\omega}(\sum_i a_i g_i) = \prod_i (\omega(a_i), (g_i)^{\text{Tr}(a_i \bmod \mathfrak{m}_i)})$. Then the sequence

$$1 \longrightarrow K_1(\mathcal{R}_G) / (\mathbb{W} \times K_1(\mathcal{R}_G)_{tors}) \xrightarrow{\mathfrak{L}_G} \mathcal{R}_G \xrightarrow{\tilde{\omega}} \mathbb{W} \times G^{ab} \longrightarrow 1$$

is exact.

Proof. The proof is a generalization of the proof of Oliver. We also use induction on the order of G to prove the result. Let $G = (1)$. Then $\mathcal{R} = \mathbb{I}[[\tilde{\Gamma}^{p^e}]]$. By [Kak13, Def 70], we have the exact sequence,

$$1 \longrightarrow \mu_K \times \mathbb{W} \longrightarrow K_1(\mathcal{R}) \xrightarrow{\mathfrak{L}} \mathcal{R} \xrightarrow{\omega} \mathbb{W} \longrightarrow 1.$$

Next, let G be a non-trivial p -group. We then show that $\tilde{\omega} \circ \mathfrak{L}_G = 1$. It is enough to prove this when G is abelian. Let I be the augmentation ideal of $\mathcal{R}[G]$. Consider $u = 1 + \sum r_i(1 - a_i)g_i \in 1 + I$, where $r_i \in \mathcal{R}$. Then

$$\begin{aligned} u^p &\equiv 1 + p \sum r_i(1 - a_i)g_i + \sum r_i^p(1 - a_i)^p g_i^p \pmod{pI^2} \\ &\equiv 1 + p \sum r_i(1 - a_i)g_i + \sum r_i\{(1 - a_i)^p - p(1 - a_i)\}g_i^p \pmod{pI^2}, \text{ by [Oli88, Lemma 6.3(i)]} \\ &\equiv \varphi_{conj}(u) + p \sum r_i(1 - a_i)(g_i - g_i^p) \pmod{pI^2} \\ &\equiv \varphi_{conj}(u) \pmod{pI^2}, \end{aligned}$$

i.e., $u^p / \varphi_{conj}(u) \in 1 + pI^2$. Therefore $\mathfrak{L}_G(u) = \log_G(u) - p^{-1} \varphi_{conj}(\log_G(u)) = \frac{1}{p} \log_G(u^p / \varphi_{conj}(u)) \in I^2$.

On the other hand, for any $r \in \mathcal{R}$ and $a, b, g \in G$, we have

$$\begin{aligned} \tilde{\omega}(r(1 - a)(1 - b)g) &= (\omega(r), g^{\text{Tr}(r)})(\omega(-r), (ag)^{\text{Tr}(-r)})(\omega(-r), (bg)^{\text{Tr}(-r)})(\omega(r), (abg)^{\text{Tr}(r)}) \\ &= (\omega(r)\omega(-r)\omega(-r)\omega(r), 1) \\ &= (1, 1) \in \mathbb{W} \times G. \end{aligned}$$

Therefore, $\mathfrak{L}_G(1 + I) \subseteq I^2 \subseteq \ker(\tilde{\omega})$, and hence

$$(104) \quad \mathfrak{L}_G(K_1(\mathcal{R}_G)) = \mathfrak{L}_G(\mathcal{R}^\times \times (1 + I)) = \langle \mathfrak{L}(\mathcal{R}^\times), \mathfrak{L}(1 + I) \rangle \subseteq \ker(\tilde{\omega}).$$

It follows that $\tilde{\omega} \circ \mathfrak{L}_G = 1$.

Assume that the theorem is true for all groups whose order is less than the order of G . Now, let z be an element of order p in the center $Z(G)$, such that z is a commutator if G is nonabelian. The existence of such a commutator is shown in [Oli88, Lemma 6.5]. Let $\hat{G} := G / \langle z \rangle$. Let $\alpha : G \longrightarrow \hat{G}$ denote the natural projection map. Then we have the following commutative

diagram:

$$\begin{array}{ccccccc}
& & 1 & & 1 & & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K_1(\mathcal{R}[G], (1-z)\mathcal{R}[G]) / \text{tors} & \xrightarrow{\mathfrak{L}_0} & \overline{H}_0(G; (1-z)\mathcal{R}[G]) & \xrightarrow{\omega_0} & \ker(\alpha^{ab}) \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K_1(\mathcal{R}[G]) / \text{tors} & \xrightarrow{\mathfrak{L}_G} & H_0(G; (1-z)\mathcal{R}[G]) & \xrightarrow{\omega_G} & \mathbb{W} \times G^{ab} \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \alpha \\
1 & \longrightarrow & K_1(\mathcal{R}[\widehat{G}]) / \text{tors} & \xrightarrow{\mathfrak{L}_{\widehat{G}}} & H_0(\widehat{G}; (1-z)\mathcal{R}[\widehat{G}]) & \xrightarrow{\omega_{\widehat{G}}} & \mathbb{W} \times \widehat{G}^{ab} \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 1 & & 1 & & 1
\end{array}$$

By [Oli88, Theorem 1.14 (iii)], the columns are all exact. By the induction hypothesis, the bottom row is exact. In the top row, the integral logarithm map \mathfrak{L}_0 is injective, by Proposition 7.18, and the map ω_0 is clearly onto. Moreover, $\text{Im}(\mathfrak{L}_0) \subset \ker(\omega_0)$, by (104). Lastly, by Proposition 7.18 again, we have,

$$\begin{aligned}
|\ker(\alpha^{ab})| &= \begin{cases} 1 & \text{if } z \text{ is a commutator} \\ p & \text{otherwise} \end{cases} \\
&= |\text{coker}(\mathfrak{L}_0)|.
\end{aligned}$$

Again, as $\omega_G \circ \mathfrak{L}_G = 1$, by the equality (104), it follows that the middle row is short exact. \square

Definition 7.20. Let $\mathbb{K} = K[[X_1, \dots, X_r, Y]]$, where K is the quotient field of \mathcal{O} , and Y is the variable corresponding to $\widetilde{\Gamma}^{p^e}$. Recall that $\mathbb{W} = (1 + p\mathbb{Z}_p)^{r+1}$. For any finite group G , we define the following groups (see [Oli88, Page 173]):

$$\begin{aligned}
SK_1(\mathcal{R}[G]) &:= \ker[K_1(\mathcal{R}[G]) \longrightarrow K_1(\mathbb{K}[G])], \\
K'_1(\mathcal{R}[G]) &:= K_1(\mathcal{R}[G]) / SK_1(\mathcal{R}[G]), \\
\text{Wh}(\mathcal{R}[G]) &:= K_1(\mathcal{R}[G]) / (\mu_K \times \mathbb{W} \times G^{ab}),
\end{aligned}$$

where μ_K is the set of roots of unity in K . This is a generalization of Definition 7.1

The following proposition is a generalization of [Oli88, Theorem 7.1].

Proposition 7.21. *Let G be a finite p -group and $z \in Z(G)$ such that the order of z is the prime p . Let*

$$\Omega = \{g \in G : [g, h] = z, \text{ for some } h \in G\}.$$

On this set, consider the following relation \sim :

$$g \sim h \text{ if } \begin{cases} g \text{ is conjugate to } h, \text{ or} \\ [g, h] = z^i, \text{ for any } i \text{ prime to } p. \end{cases}$$

Then

$$\ker[\text{tors}(\text{Wh}(\mathcal{R}[G])) \longrightarrow \text{tors}(\text{Wh}(\mathcal{R}[G/\langle z \rangle]))] \cong (\mathbb{Z}/p)^N, \text{ where } N = \begin{cases} 0 & \text{if } \Omega = \emptyset \\ |\Omega / \sim| - 1 & \text{if } \Omega \neq \emptyset. \end{cases}$$

Proof. The proof of this Proposition also follows the same argument as in the proof of Oliver. The first step is to recall the exact sequence in (103), which comes from the homomorphism

$$\log : K_1(\mathcal{R}[G], (1-z)\mathcal{R}[G]) \longrightarrow H_0(G; (1-z)\mathcal{R}[G]),$$

where $\ker(\log) = \langle z \rangle$ and $\text{im}(\log) = \{(1-z) \sum r_i g_i : r_i \in \mathcal{R}, g_i \in G, \sum r_i \in \ker(\tau)\}$. This map fits into the following commutative diagram:

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & H_0(G, (1-z)\mathcal{R}[\Omega]) \\ & & \downarrow \\ K_1(\mathcal{R}[G], (1-z)\mathcal{R}[G]) & \xrightarrow{\log} & H_0(G, (1-z)\mathcal{R}[G]) \\ \downarrow & & \downarrow \\ K_1(\mathcal{R}[G]) & \xrightarrow{\quad} & H_0(G, \mathcal{R}[G]) \\ \downarrow \eta & \nearrow \log_{\mathcal{R}[G]} & \\ \text{Wh}(\mathcal{R}[G]) & & \end{array}$$

In the above diagram, we have used the following equality

$$\begin{aligned} & \ker[H_0(G, (1-z)\mathcal{R}[G]) \longrightarrow H_0(G, \mathcal{R}[G])] \\ (105) \quad &= \langle r(1-z)g \in H_0(G, (1-z)\mathcal{R}[G]) : g \text{ is conjugate to } gz, r \in \mathcal{R}[G] \rangle \\ &= H_0(G, (1-z)\mathcal{R}[\Omega]). \end{aligned}$$

Now, consider the surjection $K_1(\mathcal{R}[G]) \xrightarrow{\eta} \text{Wh}(\mathcal{R}[G])$. Let $x \in (\text{Wh}(\mathcal{R}[G]))_{\text{tors}}$, with $\eta(y) = x$ for some $y \in K_1(\mathcal{R}[G])$. Then, as $x^n = 1$, for some n , it follows that $y^n \in \ker(\eta)$, which is finite. Therefore $y^{nr} = 1$, for some r , and $y \in K_1(\mathcal{R}[G])_{\text{tors}}$. Hence η induces a surjection $K_1(\mathcal{R}[G])_{\text{tors}} \xrightarrow{\eta} \text{Wh}(\mathcal{R}[G])_{\text{tors}}$. On the other hand, by a straight forward generalization of [Oli88, Theorem 2.9], we have $\ker(\log_{\mathcal{R}[G]} \circ \eta) = K_1(\mathcal{R}[G])_{\text{tors}}$. Combining this with the surjection $\ker(\log_{\mathcal{R}[G]} \circ \eta) \xrightarrow{\eta} \ker(\log_{\mathcal{R}[G]})$, we have $\ker(\log_{\mathcal{R}[G]}) = \text{Wh}(\mathcal{R}[G])_{\text{tors}}$.

Further, set $I := (1-z)\mathcal{R}[G]$ and consider the map $L : 1 + I \xrightarrow{\log} I \xrightarrow{\text{proj}} I/[\mathcal{R}[G], I]$. Then by a straightforward generalization of [Oli88, Theorem 2.9], we have a surjection from $\ker(L)$ to $\ker(\log_I)$. Therefore, for any $u \in 1 + (1-z)\mathcal{R}[G]$, if $\bar{u} \in \text{Wh}(\mathcal{R}[G])$ denotes the image of u , then

$$\begin{aligned} \bar{u} \in \text{Wh}(\mathcal{R}[G])_{\text{tors}} &\iff \bar{u} \in \ker(\log_{\mathcal{R}[G]}) \\ &\iff \log(u) \in \ker[H_0(G, (1-z)\mathcal{R}[G]) \longrightarrow H_0(G, \mathcal{R}[G])] = H_0(G, (1-z)\mathcal{R}[\Omega]) \end{aligned}$$

The next step is to consider the sets

$$\begin{aligned} D &= \{\xi \in \mathcal{R}(\Omega) : (1-z)\xi \in \log(1 + (1-z)\mathcal{R}[G])\} \\ C &= \{\xi \in \mathcal{R}(\Omega) : (1-z)\xi = \log(u), \text{ for some } u \in \ker(1 + (1-z)\mathcal{R}[G] \longrightarrow \text{Wh}(\mathcal{R}[G]))\} \end{aligned}$$

and show that the required kernel is equal to D/C . This is done exactly as in the proof of [Oli88, Theorem 7.1]. \square

As a consequence of the proposition, we get the following corollary.

Corollary 7.22. *Let G be a finite p -group containing an abelian subgroup $H \trianglelefteq G$ such that G/H is cyclic. Then $SK_1(\mathcal{R}[G]) = 1$.*

Proof. The proof proceeds by induction as in [Oli88, Cor 7.2]. If $G = 1$, then as \mathcal{R} is a local ring, $SK_1(\mathcal{R}[G]) = SK_1(\mathcal{R}) = 1$ ([Bas68, Cor V.9.2]). Then assume that $H \neq 1$, and choose $z \in H \cap Z(G)$ of order p . We also assume inductively that $\text{Wh}(\mathcal{R}[G/(z)])$ is torsion free. As above, we consider the set Ω and the relation \sim . By the previous result, it is enough to show that this relation is transitive on Ω . We include the short proof for convenience. Let $\Omega \neq \emptyset$. We take any $g \in \Omega$, and any $x \in G - H$, which is a generator of G/H . Let $h \in \Omega$ such that $[g, h] = z$. As G/H is cyclic, there exists i such that either gh^i or $g^i h$ lies in H . By symmetry, we may assume that $gh^i = a \in H$. Let $h = bx^j$ for some $b \in H$, then

$$z = [g, h] = [gh^i, h] = [a, bx^j] = [a, x^j] = [ax, x^j] = [ax, x^j(ax)^{-j}] = [x, x^j(ax)^{-j}],$$

where the last equality happens as $x^j(ax)^{-j} \in H$. Therefore, in Ω , we have,

$$g \sim h \sim gh^i = a \sim x^j \sim ax \sim x^j(ax)^{-j} \sim x.$$

Hence, the relation is transitive, and the result follows. \square

Theorem 7.23. *Let G be any finite p -group. Then $(K_1(\mathcal{R}[G]))_{\text{tors}} \cong \mu_K \times G^{ab} \times SK_1(\mathcal{R}[G])$.*

Proof. If G is abelian then the previous Proposition implies the result. Now let G be any p -group, then we show that the projection map

$$pr^* : K'_1(\mathcal{R}[G])_{\text{tors}} \longrightarrow K'_1(\mathcal{R}[G^{ab}])_{\text{tors}}$$

is injective. Now, we fix the group G and assume inductively that the theorem holds for all of its proper subgroups and quotients. If G is cyclic, dihedral, quaternionic or semi-dihedral, then the Proposition holds by the previous corollary.

Since the characteristic of $\mathbb{K} = K[[X_1, \dots, X_r, Y]]$ is zero, by Maschke's theorem the ring $\mathbb{K}[G]$ is semisimple. Therefore, by Wedderburn's Theorem we have the decomposition $\mathbb{K}[G] \cong \prod_{i=1}^s A_i$, for some simple $\mathbb{K}[G]$ -modules A_i and some $s \in \mathbb{N}$. Since $\mathbb{K}[G]$ contains the field K of characteristic 0, we can show as in [Roq58, Section 2], that each of the division algebras that occur in the above decomposition is isomorphic to that of a primitive, faithful representation of some subquotient of G . In other words, the endomorphism rings of the simple modules A_i are isomorphic to that of simple modules defined over $\mathbb{K}[T]$ for subgroups or subquotients T of G . Therefore, the restriction maps and the quotient maps define the following monomorphism:

$$(106) \quad \sum \text{Res}_T^G \oplus \sum \text{Proj}_{G/N}^G : K_1(\mathbb{K}[G]) \longrightarrow \bigoplus_{T \subset G, |G:T|=p} K_1(\mathbb{K}[T]) \oplus \bigoplus_{N \trianglelefteq G, |N|=p} K_1(\mathbb{K}[G/N]).$$

It follows that the corresponding homomorphism for $K'_1(\mathbb{K}[G])$ is also injective. Next, for any subgroup H of G of index p , we have the following commutative diagram, where the maps t_1, t_2 are the transfer maps and the maps $\text{Proj}_1, \text{Proj}_2$ and Proj_3 are induced by the projection maps:

$$\begin{array}{ccccc} K'_1(\mathcal{R}[G])_{\text{tors}} & \xrightarrow{\text{Proj}_1} & K'_1(\mathcal{R}[G/[H, H]])_{\text{tors}} & \xrightarrow{\text{Proj}_3} & K'_1(\mathcal{R}[G^{ab}])_{\text{tors}} \\ \downarrow t_1 & & \downarrow t_2 & & \\ K'_1(\mathcal{R}[H])_{\text{tors}} & \xrightarrow{\text{Proj}_2} & K'_1(\mathcal{R}[H^{ab}])_{\text{tors}} & & \end{array}$$

Here the map Proj_2 is injective by the induction assumption. Regarding the map Proj_3 , it is also injective by Corollary 7.22 above, since $G/[H, H]$ contains an abelian subgroup of index p . Therefore, for any $u \in \ker(\text{Proj}_3 \circ \text{Proj}_1)$, we have $t_1(u) = 1 \in K'_1(\mathcal{R}[H^{ab}])$. Together with the fact that $\text{Proj}_{G/N}^G(u) = 1$, for all $N \trianglelefteq G$ of order p , by the induction hypothesis, we have $u = 1$ by the injective map (106). Hence the map pr^* is injective. \square

As a consequence of this along with Proposition 7.19, we get the following result.

Corollary 7.24. *Let G be a finite p -group. Then we have the following exact sequence of groups:*

$$(107) \quad 1 \longrightarrow \mu_K \times \mathbb{W} \times G^{ab} \longrightarrow K'_1(\mathcal{R}[G]) \xrightarrow{\Sigma_G} \mathcal{R}[G] \longrightarrow \mathbb{W} \times G^{ab} \longrightarrow 1.$$

7.4 The Logarithm map over $\widehat{\mathbb{I}(Z)}_{(p)}$

Recall that $Z := Z(\mathcal{G})$, the center of \mathcal{G} . Let $\widehat{\mathcal{R}} = \widehat{\mathbb{I}(Z)}_{(p)}$ and $J(\widehat{\mathcal{R}})$ denote its Jacobson radical. Since \mathcal{G} is pro- p , the ring $\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau$ is a local ring and the Jacobson radical $J(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau)$ is its maximal ideal. We again consider the power series:

$$(108) \quad \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Lemma 7.25. *Let $G = \overline{\mathcal{G}}$. The ideal $J(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau)/p\widehat{\mathcal{R}}[G]^\tau$ is a nilpotent ideal of $\widehat{\mathcal{R}}[G]^\tau/p\widehat{\mathcal{R}}[G]^\tau$.*

Proof. Let $x \in J(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau)/p\widehat{\mathcal{R}}[G]^\tau$. Since $\widehat{\mathcal{R}}$ is a complete local ring, and G is pro- p , the maximal ideal $J(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau) = \langle \mathfrak{m}_{\widehat{\mathcal{R}}}, I_G \rangle$, where $\mathfrak{m}_{\widehat{\mathcal{R}}}$ is the maximal ideal of $\widehat{\mathcal{R}}$ and I_G is the augmentation ideal of $\widehat{\mathcal{R}}[G]^\tau$.

Let $|G| = p^r$. Then $(g-1)^{p^r} \in p\widehat{\mathcal{R}}[G]^\tau$, for any $g \in G$. Therefore, for any $x \in J(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau) = \langle \mathfrak{m}_{\widehat{\mathcal{R}}}, I_G \rangle$, we have $x^{p^r} \in \langle p, \mathfrak{m}_{\widehat{\mathcal{R}}} \rangle$. Hence $x^n \in \langle p, \mathfrak{m}_{\widehat{\mathcal{R}}} \rangle^m \widehat{\mathcal{R}}[G]^\tau$ for large enough n, m . This implies that x^i/i converges to 0 as i tends to infinity. Hence the series $\text{Log}(1+x)$ converges in $\widehat{\mathcal{R}}[G]^\tau[\frac{1}{p}]$. \square

The techniques of Oliver in [Oli88, Lemma 27] as generalized by Kakde to prove [Kak13, Lemma 66], can further be generalized easily to show the following Lemma.

Lemma 7.26. *Let $I \subset J(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau)$ be any ideal of $\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau$. Then*

(i) *For any $x, y \in I$, the series $\text{Log}(1+x)$ converges to an element in $\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau[\frac{1}{p}]$, and*

$$(109) \quad \text{Log}((1+x)(1+y)) \equiv \text{Log}(1+x) + \text{Log}(1+y) \pmod{[\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau[\frac{1}{p}], I[\frac{1}{p}]]}$$

(ii) *If $I \subset \xi \widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau$, for some central element ξ such that $\xi^p \in p\xi \widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau$, then for any $x, y \in I$, $\text{Log}(1+x)$ and $\text{Log}(1+y)$ converge in I , and*

$$(110) \quad \text{Log}((1+x)(1+y)) \equiv \text{Log}(1+x) + \text{Log}(1+y) \pmod{[\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau, I]}.$$

Moreover, if $I^p \subset pIJ(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau)$, then

(a) *for all $x \in I$ the series $\text{Exp}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to an element in $1+I$.*

(b) *the maps Log and Exp are bijections and inverse to each other between $1+I$ and I .*

Proposition 7.27. *Let I be any ideal contained in the maximal ideal $J(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau)$, then the logarithm map*

$$\text{Log}(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

defined on I , induces a unique homomorphism

$$\log_I : K_1(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau, I) \longrightarrow \left(\frac{I}{[\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau, I]} \right) \left[\frac{1}{p} \right].$$

If, in addition, $I \subset \xi \widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau$, for some central element ξ such that $\xi^p \in p\xi \widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau$, then the logarithm map Log induces a homomorphism

$$\log_I : K_1(\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau, I) \longrightarrow \frac{I}{[\widehat{\mathcal{R}}[\overline{\mathcal{G}}]^\tau, I]}.$$

Proof. The proof is the same as the proof of [Kak13, Prop 67]. \square

7.5 The Integral Logarithm over $\widehat{\mathbb{I}(Z)}_{(p)}$

We now define the integral logarithm over the ring $\widehat{\mathbb{I}(Z)}_{(p)}$. For this we first consider the kernel

$$J := \ker \left[\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}} \longrightarrow \widehat{\mathbb{I}(\Gamma)_{(p)}} \right].$$

Note that the ring $\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}$ is local, therefore we have the surjective map

$$\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}^{\times} \longrightarrow K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}) \text{ and } 1 + J \longrightarrow K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}, J).$$

Now consider the exact sequence of groups which is split by the embedding $\Gamma \hookrightarrow \mathcal{G}$,

$$1 \longrightarrow 1 + J \longrightarrow \widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}^{\times} \longrightarrow \widehat{\mathbb{I}(\Gamma)_{(p)}}^{\times} \longrightarrow 1.$$

It is easy to see that any $x \in \widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}^{\times}$ can be expressed uniquely as $x = uy$, where $u \in 1 + J$ and $y \in \widehat{\mathbb{I}(\Gamma)_{(p)}}$. Hence, any $x \in K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}})$ can be written uniquely as a product $x = uy$, where $y \in K'_1(\widehat{\mathbb{I}(\Gamma)_{(p)}})$ and u lies in the image of $K_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}, J)$ in $K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}})$. We record this fact below.

Lemma 7.28. *Any $x \in K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}})$ can be written uniquely as a product $x = uy$, where $y \in K'_1(\widehat{\mathbb{I}(\Gamma)_{(p)}})$ and u lies in the image of $K_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}, J)$ in $K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}})$.*

Recall the following assumption

(Ur) $\mathbb{I} = \mathcal{O}[[X_1, \dots, X_r]]$ with \mathcal{O} unramified over \mathbb{Z}_p .

Therefore, $\widehat{\mathbb{I}(\Gamma)_{(p)}}/p\widehat{\mathbb{I}(\Gamma)_{(p)}} \cong \mathbb{F}_q[[X_1, \dots, X_r]][[\Gamma]]$, where \mathbb{F}_q is the finite field of order q and characteristic p . Consider the map

$$\varphi : \mathbb{F}_q[[X_1, \dots, X_r]][[\Gamma]] \longrightarrow \mathbb{F}_q[[X_1, \dots, X_r]][[\Gamma]]$$

such that its restriction to \mathcal{O} is the Frobenius and maps X_j^n to X_j^{pn} for all $j = 1, \dots, r$ and the p -th power map on Γ .

Lemma 7.29. *Let $y \in K_1(\widehat{\mathbb{I}(\Gamma)_{(p)}})$. Then*

$$\frac{y^p}{\varphi(y)} \equiv 1 \pmod{p\widehat{\mathbb{I}(\Gamma)_{(p)}}}.$$

Therefore, $\text{Log}(\frac{y^p}{\varphi(y)})$ is defined.

Proof. It is enough to show the congruence and this follows by computing y^p . For this, let $\bar{y} \in \mathbb{F}_q[[X_1, \dots, X_r]][[\Gamma]] \cong \mathbb{F}_q[[X_1, \dots, X_r]][[T]]$. Then $\bar{y} = \sum_{i=0}^{\infty} a_i T^i$, with $a_i \in \mathbb{F}_q[[X_1, \dots, X_r]]$. Then $\bar{y}^p = (\sum_{i=0}^{\infty} a_i T^i)^p = \sum_{i=0}^{\infty} a_i^p T^{ip} = \varphi(\bar{y})$. Therefore $\frac{y^p}{\varphi(y)} \equiv 1 \pmod{p\widehat{\mathbb{I}(\Gamma)_{(p)}}}$. \square

Definition 7.30. Let $x \in K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}})$. Then $x = uy$, with $y \in K'_1(\widehat{\mathbb{I}(\Gamma)_{(p)}})$ and u lies in the image of $K_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}, J)$ in $K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}})$. We define the *integral logarithm* map on $K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}})$ by

$$L(x) = L(uy) = L(u) + L(y) = \text{Log}(u) - \frac{1}{p}\varphi(\text{Log}(u)) + \frac{1}{p}\text{Log}\left(\frac{y^p}{\varphi(y)}\right).$$

For this integral logarithm, we have the following result which is proven exactly as in [Kak13, Prop 74].

Proposition 7.31. *The integral logarithm map defined above induces a homomorphism*

$$L : K'_1(\widehat{\mathbb{I}[[\mathcal{G}]]_{\mathcal{S}}}) \longrightarrow \widehat{\mathbb{I}(Z)}_{(p)}[\text{Conj}(\bar{\mathcal{G}})]^{\tau},$$

which is independent of the choice of the splitting of $\mathcal{G} \rightarrow \Gamma$.

7.6 The Logarithm map under restriction maps

For any $P \leq \overline{\mathcal{G}}$, recall the map

$$t_P^{\overline{\mathcal{G}}} : \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau \longrightarrow \mathcal{R}[P^{ab}]^\tau$$

defined by

$$t_P^{\overline{\mathcal{G}}}(\bar{g}) = \sum_{x \in C(\overline{\mathcal{G}}, P)} \{(\bar{x}^{-1})(\bar{g})(\bar{x}) \mid x^{-1}gx \in P\},$$

where $C(\overline{\mathcal{G}}, P)$ is the set of left coset representatives of P in $\overline{\mathcal{G}}$.

Let $\mathbb{K} = K[[X_1, \dots, X_r]]$. Then the map $t_P^{\overline{\mathcal{G}}}$ can be naturally extended to a map

$$t_P^{\overline{\mathcal{G}}} : \mathbb{K}[[Z]][\text{Conj}(\overline{\mathcal{G}})]^\tau \longrightarrow \mathbb{K}[[Z]][\text{Conj}(P)]^\tau.$$

Lemma 7.32. *For any $P \leq \overline{\mathcal{G}}$, we have the commutative diagram*

$$\begin{array}{ccc} K'_1(\mathbb{I}[[\mathcal{G}]]) & \xrightarrow{\log} & \mathbb{K}[[Z]][\text{Conj}(\overline{\mathcal{G}})]^\tau \\ \Theta_P^{\overline{\mathcal{G}}} \downarrow & & \downarrow t_P^{\overline{\mathcal{G}}} \\ K'_1(\mathbb{I}(U_P)) & \xrightarrow{\log} & \mathbb{K}[[Z]][\text{Conj}(P)]^\tau \end{array}$$

Similarly, for $J = \ker [\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}} \longrightarrow \widehat{\mathbb{I}(\Gamma)_{(p)}}]$, we also have

$$\begin{array}{ccc} K_1\left(\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}}, J\right) & \xrightarrow{\text{Log}} & \widehat{\mathbb{I}(Z)_{(p)}}[\text{Conj}(\overline{\mathcal{G}})]^\tau [\tfrac{1}{p}] \\ \Theta_P^{\overline{\mathcal{G}}} \downarrow & & \downarrow t_P^{\overline{\mathcal{G}}} \\ K_1\left(\widehat{\mathbb{I}(U_P)}, J\right) & \xrightarrow{\text{Log}} & \widehat{\mathbb{I}(Z)_{(p)}}[\text{Conj}(P)]^\tau [\tfrac{1}{p}]. \end{array}$$

The proof of this lemma proceeds exactly as in [Oli88, Theorem 6.8] and for the second part consider any $u \in K_1\left(\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}}, J\right)$, then the commutativity follows from the following equalities

$$\begin{aligned} \text{Log}(u) &= \lim_{n \rightarrow \infty} \frac{1}{p^n} (u^{p^n} - 1) \\ \Theta_P^{\overline{\mathcal{G}}}(u) &= \lim_{n \rightarrow \infty} (1 + t_P^{\overline{\mathcal{G}}}(u^{p^n} - 1))^{1/p^n}. \end{aligned}$$

Next, proceeding as in the proof of [Oli88, Lemma 77], we get the following commutative diagram.

Lemma 7.33. *Let $P \in C(\overline{\mathcal{G}})$ be a non-trivial subgroup. Then, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{I}(U_P)^\times & \xrightarrow{\log} & \mathbb{K}[U_P] \\ \alpha_P \downarrow & & \downarrow p\eta_P \\ \mathbb{I}(U_P)^\times & \xrightarrow{\log} & \mathbb{K}[U_P] \end{array}$$

Similarly, the following diagram is also commutative

$$\begin{array}{ccc} K_1\left(\widehat{\mathbb{I}(U_P)}_{\mathcal{S}}, J\right) & \xrightarrow{\text{Log}} & \widehat{\mathbb{I}[U_P]}_{\mathcal{S}} [\tfrac{1}{p}] \\ \alpha_P \downarrow & & \downarrow p\eta_P \\ K_1\left(\widehat{\mathbb{I}(U_P)}, J\right) & \xrightarrow{\text{Log}} & \widehat{\mathbb{I}[U_P]}_{\mathcal{S}} [\tfrac{1}{p}]. \end{array}$$

To establish a compatibility between the subgroups, we also consider the following maps.

Definition 7.34. Define a map $v_P^{\overline{\mathcal{G}}} : \prod_{C \leq \overline{\mathcal{G}}} \mathbb{K}[[Z]][C^{ab}] \longrightarrow \mathbb{K}[[Z]][P^{ab}]$, as follows:

If P is not cyclic, then

$$v_P^{\overline{\mathcal{G}}}((x_C)) = \left(\sum_{P'} \frac{[P : P']}{P : (P')^p} \varphi(x_{P'}) \right)$$

where P' runs over all subgroups contained in $C(\overline{\mathcal{G}})$ such that $(P')^p \leq P$.

If P is cyclic, then

$$v_P^{\overline{\mathcal{G}}}((x_C)) = \sum_{P'} [P : (P')^p] \varphi(x_{P'}) = p \sum_{P'} \varphi(x_{P'}),$$

where P' runs over all the $P' \in C(\overline{\mathcal{G}})$ with $(P')^p = P$ but $P' \neq P$.

We set $v^{\overline{\mathcal{G}}} = (v_P^{\overline{\mathcal{G}}})_P$.

Analogously, we define maps in the case of the p -adic completions, and we denote them again by $v^{\overline{\mathcal{G}}}$.

Then we can show the following lemma as in [Kak13, Lemma 79].

Lemma 7.35. *Let $P \neq 1$. Then the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{K}[[Z]][\text{Conj}(\overline{\mathcal{G}})]^\tau & \xrightarrow{\varphi} & \mathbb{K}[[Z]][\text{Conj}(\overline{\mathcal{G}})]^\tau \\ \beta^{\overline{\mathcal{G}}} \downarrow & & \downarrow \beta_P^{\overline{\mathcal{G}}} \\ \prod_{C \leq \overline{\mathcal{G}}} \mathbb{K}[[Z]][C^{ab}] & \xrightarrow{v_P^{\overline{\mathcal{G}}}} & \mathbb{K}[[Z]][P^{ab}]. \end{array}$$

Let $P = \{1\}$, then we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{K}[[Z]][\text{Conj}(\overline{\mathcal{G}})]^\tau & \xrightarrow{\varphi} & \mathbb{K}[[Z]][\text{Conj}(\overline{\mathcal{G}})]^\tau \\ \beta^{\overline{\mathcal{G}}} \downarrow & & \downarrow \beta_P^{\overline{\mathcal{G}}} \\ \prod_{C \leq \overline{\mathcal{G}}} \mathbb{K}[[Z]][C^{ab}] & \xrightarrow{\varphi + v_1^{\overline{\mathcal{G}}}} & \mathbb{K}[[Z]]. \end{array}$$

Analogous results hold for the p -adic completions.

Definition 7.36. Define the map $u_P^{\overline{\mathcal{G}}} : \prod_{C \leq \overline{\mathcal{G}}} \mathbb{I}(U_C^{ab})^\times \longrightarrow \mathbb{I}(U_P^{ab})^\times$ as follows:

If P is not a cyclic subgroup of $\overline{\mathcal{G}}$, then

$$u_P^{\overline{\mathcal{G}}}((x_C)) = \prod_{P'} \varphi(x_{P'})^{|P'|},$$

where P' runs over all subgroups contained in $C(\overline{\mathcal{G}})$ such that $(P')^p \leq P$.

If P is cyclic, then

$$u_P^{\overline{\mathcal{G}}}((x_C)) = \prod_{P'} \varphi(x_{P'}),$$

where P' runs over all the $P' \in C(\overline{\mathcal{G}})$ with $(P')^p = P$ but $P' \neq P$.

Then, we define the collection of maps by $u^{\overline{\mathcal{G}}} = (u_P^{\overline{\mathcal{G}}})_P$. Analogously, we define maps in the case of the p -adic completions, and we denote them again by $v^{\overline{\mathcal{G}}}$.

As the logarithm maps that we have defined respects group homomorphisms of Iwasawa algebras induced by the group homomorphism, we have the following lemma.

Lemma 7.37. *Let P be a non-cyclic subgroup of $\overline{\mathcal{G}}$. Then the following diagram is commutative.*

$$\begin{array}{ccc} \prod_{C \leq \overline{\mathcal{G}}} \mathbb{I}(U_C^{ab})^\times & \xrightarrow{\log} & \prod_{C \leq \overline{\mathcal{G}}} \mathbb{K}[[Z]][C^{ab}] \\ u_P^{\overline{\mathcal{G}}} \downarrow & & \downarrow |P|v_P^{\overline{\mathcal{G}}} \\ \mathbb{I}(U_P^{ab})^\times & \xrightarrow{\log} & \mathbb{K}[[Z]][P^{ab}]. \end{array}$$

Let P be a cyclic subgroup of $\overline{\mathcal{G}}$. Then the following diagram is commutative.

$$\begin{array}{ccc} \prod_{C \leq \overline{\mathcal{G}}} \mathbb{I}(U_C^{ab})^\times & \xrightarrow{\log} & \prod_{C \leq \overline{\mathcal{G}}} \mathbb{K}[[Z]][C^{ab}] \\ u_P^{\overline{\mathcal{G}}} \downarrow & & \downarrow \frac{1}{p}v_P^{\overline{\mathcal{G}}} \\ \mathbb{I}(U_P^{ab})^\times & \xrightarrow{\log} & \mathbb{K}[[Z]][P^{ab}]. \end{array}$$

Recall that $\widehat{\mathcal{H}} = \widehat{\mathbb{I}(Z)_{(p)}}$ and let $J_P = \ker \left[\widehat{\mathcal{H}}[P^{ab}] \rightarrow \widehat{\mathbb{I}(\Gamma)_{(p)}} \right]$.

Let P be a non-cyclic subgroup of $\overline{\mathcal{G}}$. Then we have the following commutative diagram.

$$\begin{array}{ccc} \prod_{C \leq \overline{\mathcal{G}}} 1 + J_C & \xrightarrow{\text{Log}} & \prod_{C \leq \overline{\mathcal{G}}} \mathbb{Q}_p \otimes J_C \\ U_P^{\overline{\mathcal{G}}} \downarrow & & \downarrow |P|v_P^{\overline{\mathcal{G}}} \\ 1 + J_P & \xrightarrow{\text{Log}} & \mathbb{Q}_p \otimes J_P. \end{array}$$

Let P be a cyclic subgroup of $\overline{\mathcal{G}}$. Then the following diagram is commutative.

$$\begin{array}{ccc} \prod_{C \leq \overline{\mathcal{G}}} 1 + J_C & \xrightarrow{\text{Log}} & \prod_{C \leq \overline{\mathcal{G}}} \mathbb{Q}_p \otimes J_C \\ U_P^{\overline{\mathcal{G}}} \downarrow & & \downarrow \frac{1}{p}v_P^{\overline{\mathcal{G}}} \\ 1 + J_P & \xrightarrow{\text{Log}} & \mathbb{Q}_p \otimes J_P. \end{array}$$

Lemma 7.38. *Let $x \in K'_1(\mathcal{H}[\overline{\mathcal{G}}]^\tau)$ or $K'_1(\widehat{\mathcal{H}}[\overline{\mathcal{G}}]^\tau)$. Then for every non-cyclic subgroup $P \leq \overline{\mathcal{G}}$, we have*

$$\alpha_P(\theta_P^{\overline{\mathcal{G}}}(x))^{p|P|} \equiv u_P^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x))) \pmod{p}.$$

In particular, the logarithm $\log \left(\frac{\alpha_P(\theta_P^{\overline{\mathcal{G}}}(x))^{p|P|}}{u_P^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x)))} \right)$ is well-defined.

Proof. Let C be cyclic. Then $\alpha_C(\theta_C^{\overline{\mathcal{G}}}(x)) \equiv 1 \pmod{p}$ and $u_P^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x))) \equiv \varphi(\alpha_{\{1\}})(\Theta_{\{1\}}^{\overline{\mathcal{G}}}) \pmod{p}$. Then the result follows from the congruence $\theta_P^{\overline{\mathcal{G}}}(x)^{|P|} \equiv \theta_{\{1\}}^{\overline{\mathcal{G}}}(x) \pmod{p}$, which follows from a straightforward generalization of [SV13, Prop 2.3]. \square

We now give the relation between the multiplicative and the additive sides. For completeness and to see how the lemmas proved above are used we also give a proof of one of the formulas on the lines of [Kak13, Prop 84].

Proposition 7.39. *Let $x \in K'_1(\mathbb{I}[[\mathcal{G}]])$. Then*

$$\beta_P^{\overline{\mathcal{G}}}(L(x)) = \begin{cases} \frac{1}{p^2|P|} \log \left(\frac{\alpha_P(\theta_P^{\overline{\mathcal{G}}}(x))^{p|P|}}{u_P^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x)))} \right), & \text{if } P \notin C(\overline{\mathcal{G}}) \\ \frac{1}{p} \log \left(\frac{\alpha_P(\theta_P^{\overline{\mathcal{G}}}(x))^{p|P|}}{u_P^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x)))} \right), & \text{if } P \in C(\overline{\mathcal{G}}), P \neq \{1\} \\ \frac{1}{p} \log \left(\frac{\alpha_{\{1\}}(\theta_{\{1\}}^{\overline{\mathcal{G}}}(x))^p}{\varphi(\theta_{\{1\}}^{\overline{\mathcal{G}}}(x))(u_{\{1\}}^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x)))} \right), & \text{if } P = \{1\}. \end{cases}$$

We also have analogous relations over the p -adic completions $K'_1(\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}})$.

Proof. Let $x \in K'_1(\mathbb{I}[[\mathcal{G}]])$. In the first case, we consider a group $P \in C(\overline{\mathcal{G}})$. Then we have

$$\begin{aligned}
\beta_P^{\overline{\mathcal{G}}}(L(x)) &= \beta_P^{\overline{\mathcal{G}}}(\log(x) - \frac{\varphi}{p}(\log(x))) \\
&= \frac{1}{p} \log(\alpha_P(\Theta_P^{\overline{\mathcal{G}}}(x))) - \beta_P^{\overline{\mathcal{G}}}(\frac{\varphi}{p}(\log(x))), & (\text{Lemmas 7.32, 7.33}) \\
&= \frac{1}{p} \log(\alpha_P(\Theta_P^{\overline{\mathcal{G}}}(x))) - \frac{1}{p} v_P^{\overline{\mathcal{G}}}(\beta_P^{\overline{\mathcal{G}}}(\log(x))), & (\text{Lemma 7.35}) \\
&= \frac{1}{p} \log(\alpha_P(\Theta_P^{\overline{\mathcal{G}}}(x))) - \frac{1}{p^2} v_P^{\overline{\mathcal{G}}}(\log(\alpha(\Theta_P^{\overline{\mathcal{G}}}(x)))), & (\text{Lemmas 7.32, 7.33}) \\
&= \frac{1}{p} \log(\alpha_P(\Theta_P^{\overline{\mathcal{G}}}(x))) - \frac{1}{p^2 |P|} \log(u_P^{\overline{\mathcal{G}}}(\alpha(\Theta_P^{\overline{\mathcal{G}}}(x)))), & (\text{Lemma 7.37}) \\
&= \frac{1}{p^2 |P|} \log \left(\frac{\alpha_P(\Theta_P^{\overline{\mathcal{G}}}(x))^{p|P|}}{u_P^{\overline{\mathcal{G}}}(\alpha(\Theta_P^{\overline{\mathcal{G}}}(x)))} \right)
\end{aligned}$$

If $P \in C(\overline{\mathcal{G}})$, $P \neq \{1\}$, or $P = \{1\}$, then the formula for $\beta_P^{\overline{\mathcal{G}}}$ can also be shown similarly as in [Kak13, Prop 84].

We now give a proof if $x \in K'_1(\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}})$. For this, we first note that we can write $x = uy$, for some $u \in \text{im} \left[K_1 \left(\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}}, J \right) \rightarrow K'_1(\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}}) \right]$ and $y \in K'_1(\widehat{\mathbb{I}(\Gamma)}_{(p)})$. (Recall that $J = \ker[\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}} \rightarrow \widehat{\mathbb{I}(\Gamma)}_{(p)}]$). The proof for u is the same as above, and in fact, we may show that it is true for any u in the image of $K_1 \left(\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}}, J_{\mathcal{R}} \right)$, where $\widehat{J}_{\mathcal{R}}$ is the Jacobson radical of $\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}}$.

We now show it for $y \in \widehat{\mathbb{I}(Z)}_{(p)}$. Note that $\Theta_P^{\overline{\mathcal{G}}}(y) = y^{[\overline{\mathcal{G}}:P]}$, for all $P \in \overline{\mathcal{G}}$. Further, if P is a non-trivial cyclic subgroup of $\overline{\mathcal{G}}$, then $\alpha_P(\Theta_P^{\overline{\mathcal{G}}}(y)) = 1$. On the other hand, since $L(y) \in \widehat{\mathbb{I}(Z)}_{(p)}$, we have

$$\beta_P^{\overline{\mathcal{G}}}(L(y)) = \begin{cases} [\overline{\mathcal{G}} : P] L(y), & \text{if } P \text{ is noncyclic or } P = \{1\} \\ 0, & \text{if } P \text{ is cyclic.} \end{cases}$$

Now, it is easy to see that the formula for $\beta_P^{\overline{\mathcal{G}}}(L(y))$ holds. We now consider the case when $y \in \widehat{\mathbb{I}(\Gamma)}_{(p)}$. In this case, $\varphi(y)^r \in \widehat{\mathbb{I}(Z)}_{(p)}$, for some r . Then $\frac{\varphi(y)^r}{y^{p^r}} \equiv 1 \pmod{\widehat{\mathbb{I}(\Gamma)}_{(p)}}$ and hence in the image of $K_1 \left(\widehat{\mathbb{I}[[\mathcal{G}]}}_{\mathcal{S}}, J_{\mathcal{R}} \right)$. Therefore the formula holds for $\frac{\varphi(y)^r}{y^{p^r}}$ and also $\varphi^r(y)$. Hence, the formula holds for y^{p^r} . Since the image of $\beta_P^{\overline{\mathcal{G}}}$ is torsion free abelian group, therefore the formula holds for $\beta_P^{\overline{\mathcal{G}}}$. \square

7.7 Congruences over $\mathbb{I}[[\mathcal{G}]]$

For any subgroup P of $\overline{\mathcal{G}}$, we write $\Theta_P^{\overline{\mathcal{G}}, ab}$ for the following natural composite homomorphism

$$K'_1(\mathbb{I}[[\mathcal{G}]]) \xrightarrow{\Theta_P^{\overline{\mathcal{G}}}} K'_1(\mathbb{I}(U_P)) \rightarrow K_1(\mathbb{I}(U_P^{ab})) \cong \mathbb{I}(U_P^{ab})^\times,$$

where the isomorphism is induced by taking determinants over $\mathbb{I}(U_P^{ab})$. We now show that the image of the map $\Theta^{\overline{\mathcal{G}}} = (\Theta_P^{\overline{\mathcal{G}}})$ lies in $\Phi^{\mathcal{G}}$.

Theorem 7.40. *Let $\Xi \in K'_1(\mathbb{I}[[\mathcal{G}]])$ and for all subgroups P of $\overline{\mathcal{G}}$, put $\Xi_{U_P^{ab}} := \Theta_P^{\overline{\mathcal{G}}, ab}(\Xi) \in \mathbb{I}(U_P^{ab})^\times$.*

(i) *For all subgroups P, P' of $\overline{\mathcal{G}}$ with $[P', P'] \leq P \leq P'$, we have*

$$\text{Nr}_{P'}^{P'}(\Xi_{U_{P'}^{ab}}) = \Pi_P^{P'}(\Xi_{U_P^{ab}}).$$

(ii) For all subgroups P of $\overline{\mathcal{G}}$ and all g in $\overline{\mathcal{G}}$ we have $\Xi_{gU_P^{ab}g^{-1}} = g\Xi_{U_P^{ab}}g^{-1}$.

(iii) For every $P \in \overline{\mathcal{G}}$ and $P \neq (1)$, we have

$$\text{ver}_P^{P'}(\Xi_{U_P^{ab}}) \equiv \Xi_{U_P^{ab}} \pmod{\mathcal{T}_{P,P'}} \text{ (resp. } \mathcal{T}_{P,P',\mathcal{S}} \text{ and } \widehat{\mathcal{T}}_{P,P'}).$$

(iv) For all $P \in C(\overline{\mathcal{G}})$ we have $\alpha_P(\Xi_{U_P^{ab}}) \equiv \prod_{P' \in C_P(\overline{\mathcal{G}})} \alpha_{P'}(\Xi_{U_{P'}^{ab}}) \pmod{p\mathcal{T}_P}$.

To prove this theorem, we recall an explicit description of the map $\Theta_P^{P',ab}$. We write $n_{P'/P} := [P' : P] = [U_{P'} : U_P]$. Since $\mathbb{I}(U_{P'})$ is a local ring, the natural homomorphism

$$q_{P'} : \mathbb{I}(U_{P'})^\times \longrightarrow K_1(\mathbb{I}(U_{P'}))$$

is *surjective*. For any $\Xi \in K_1(\mathbb{I}(U_{P'}))$, let $\tilde{\Xi} \in \mathbb{I}(U_{P'})^\times$ denote a pre-image under $q_{P'}$. We denote the set of left coset representatives of U_P in $U_{P'}$ by $C(P', P) := \{c_i : 1 \leq i \leq n_{P'/P}\}$. Then as an $\mathbb{I}(U_{P'})$ -module we have

$$\mathbb{I}(U_{P'}) \cong \bigoplus_{i=1}^{n_{P'/P}} \mathbb{I}(U_P)c_i.$$

Let $M_{C(P',P)}(\tilde{\Xi})$ denote the matrix in $M_{n_{P'/P}}(\mathbb{I}(U_P))$ of the automorphism given by multiplication by $\tilde{\Xi}$ on the right, and $\Pi_{P',P} : M_{n_{P'/P}}(\mathbb{I}(U_P)) \longrightarrow M_{n_{P'/P}}(\mathbb{I}(U_P^{ab}))$ denote the natural projection. Then

$$\Theta_P^{P',ab}(\Xi) = \det \left(\Pi_{P',P}(M_{C(P',P)}(\tilde{\Xi})) \right) \in \mathbb{I}(U_P^{ab})^\times,$$

Proof of Theorem 7.40(i): Consider the following diagram

$$\begin{array}{ccc} K'_1(\mathbb{I}[[\mathcal{G}]]) & \xrightarrow{\Theta_P^{\overline{\mathcal{G}}}} & K'_1(\mathbb{I}(U_P)) \\ \downarrow \Theta_{P'}^{\overline{\mathcal{G}}} & & \downarrow \pi_P \\ K'_1(\mathbb{I}(U_{P'})) & \xrightarrow{\Theta_P^{P'}} & \mathbb{I}(U_P^{ab})^\times \\ \downarrow \pi_{P'} & & \downarrow \Pi_P^{P'} \\ \mathbb{I}(U_{P'})^\times & \searrow \text{Nr}_P^{P'} & \mathbb{I}(U_P/[U_{P'}, U_{P'}])^\times \end{array}$$

The upper quadrilateral in the diagram is obviously seen to be commutative. The lower quadrilateral is also commutative since the coset space $C(P', P)$ can be regarded as an $\mathbb{I}(U_P/[U_{P'}, U_{P'}])$ -basis of $\mathbb{I}(U_{P'}^{ab})$. Therefore, we have

$$\text{Nr}_P^{P'}(\Xi_{P'}) = \text{Nr}_P^{P'}(\pi_{P'}(\tilde{\Xi})) = \Pi_P^{P'}(\det(\Pi_{P',P}(M_{C(P',P)}(\tilde{\Xi}))) = \Pi_P^{P'}(\Theta_P^{P',ab}(\Xi)).$$

□

Proof of Theorem 7.40(ii): Let $C := C(P, \overline{\mathcal{G}})$. Then, for any $g \in \overline{\mathcal{G}}$, the set $gCg^{-1} := \{gc_i g^{-1} \mid c_i \in C\}$ is a set of left coset representatives of $gU_P g^{-1} = U_{gPg^{-1}}$ in \mathcal{G} . By definition, we have

$$\Xi_{gPg^{-1}} = \Theta_P^{\overline{\mathcal{G}},ab}(\Xi) = \det \left(\Pi_{\overline{\mathcal{G}},gPg^{-1}}(gM_C(\tilde{\Xi})g^{-1}) \right) = g \det \left(\Pi_{\overline{\mathcal{G}},P}(M_C(\tilde{\Xi})) \right) g^{-1} = g\Xi_P g^{-1}.$$

The equality follows from this. □

Proof of Theorem 7.40(iii): The proof of (C3) is same as the proof of (M3) in [Kak13, Lemma 85]. \square

For the proof of Theorem 7.40(iv), we need the following lemma.

Lemma 7.41. *For all $x \in K'_1(\mathbb{I}[[\mathcal{G}]])$ and all $P \in C(\overline{\mathcal{G}})$, we have,*

$$\log_P \left(\frac{\alpha_P(\Theta_P^{\mathcal{G}}(x))}{\prod_{P' \in C_P(\overline{\mathcal{G}})} \alpha_{P'}(\Theta_P^{\mathcal{G}}(x))} \right) = p(\eta_P \circ \text{Res}_P^{\overline{\mathcal{G}}})(\mathfrak{L}_{\overline{\mathcal{G}}}(x)).$$

Proof. The lemma follows from the commutativity of the following diagram

$$\begin{array}{ccc} K'_1(\mathbb{I}[[\mathcal{G}]]) & \xrightarrow{\log_{\overline{\mathcal{G}}}} & \mathbb{I}[\text{Conj}(\overline{\mathcal{G}})]^{\tau}[\frac{1}{p}] \\ \downarrow \Theta_P^{\overline{\mathcal{G}}} & & \downarrow \text{Res}_P^{\overline{\mathcal{G}}} \\ \prod_{P \in C(\overline{\mathcal{G}})} K'_1(\mathbb{I}(U_P)) & \xrightarrow{\log_P} & \prod_{P \in C(\overline{\mathcal{G}})} \mathbb{I}[P]^{\tau}[\frac{1}{p}] \\ & \searrow b & \nearrow c \\ & & \prod_{P \in C(\overline{\mathcal{G}})} \mathbb{I}[P]^{\tau}[\frac{1}{p}], \end{array}$$

where a is a curved arrow from $\mathbb{I}[\text{Conj}(\overline{\mathcal{G}})]^{\tau}[\frac{1}{p}]$ to $\prod_{P \in C(\overline{\mathcal{G}})} \mathbb{I}[P]^{\tau}[\frac{1}{p}]$, and b is a curved arrow from $\prod_{P \in C(\overline{\mathcal{G}})} K'_1(\mathbb{I}(U_P))$ to $\prod_{P \in C(\overline{\mathcal{G}})} \mathbb{I}[P]^{\tau}[\frac{1}{p}]$.

where

$$\begin{aligned} a(x) &:= \left(\eta_P(\text{Res}_P^{\overline{\mathcal{G}}}((1 - p^{-1}\varphi_{\text{conj}})(x))) \right)_P \\ b((x_P)_P) &:= \left(p^{-1} \log_Q \left(\frac{\alpha_Q(\Theta_Q^{\mathcal{G}}(x_Q))}{\prod_{P' \in C_Q(\overline{\mathcal{G}})} \alpha_{P'}(\Theta_P^{\mathcal{G}}(x_{P'}))} \right) \right)_Q. \end{aligned}$$

By Lemma 7.15, the square in the diagram is commutative. To show that the triangles commute, as in [Kak13], the map c is chosen as

$$c((x_P)_P) := \left(1 - \delta_P p^{-1} \varphi_{\text{conj}}(x_P) - \sum_{P' \in C_P(\overline{\mathcal{G}})} \varphi_{\text{conj}}(x_{P'}) \right)_P,$$

where $\delta_P = 1$ if P is non-trivial, and 0 otherwise. Then the commutativity of the triangles follow analogously as in [Kak13, Lemma 7.4]. \square

Lemma 7.42. *We have $\text{Res}_P^{\overline{\mathcal{G}}}(\mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^{\tau}) \subset \mathcal{T}_P$.*

Proof. Let $x \in \mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^{\tau}$. Then $x := \sum_{(i_1, \dots, i_n) \geq 0} (\sum_{g \in \overline{\mathcal{G}}} c_{g, (i_1, \dots, i_n)} \kappa(g)) X_1^{i_1} \dots X_n^{i_n}$, where $c_{g, (i_1, \dots, i_n)} \in \mathcal{R}$, for all g and (i_1, \dots, i_n) . Consider the normalizer $N_{\overline{\mathcal{G}}}(P)$ of P in $\overline{\mathcal{G}}$. Then $\text{Res}_P^{\overline{\mathcal{G}}} = \text{Res}_P^{N_{\overline{\mathcal{G}}}(P)} \circ \text{Res}_{N_{\overline{\mathcal{G}}}(P)}^{\overline{\mathcal{G}}}$. Therefore

$$\text{Res}_P^{\overline{\mathcal{G}}}(x) = \sum_{(i_1, \dots, i_n) \geq 0} \left(\sum_{g \in \overline{\mathcal{G}}} c_{g, (i_1, \dots, i_n)} \text{Res}_P^{N_{\overline{\mathcal{G}}}(P)} \left(\text{Res}_{N_{\overline{\mathcal{G}}}(P)}^{\overline{\mathcal{G}}}(\kappa(g)) \right) \right) X_1^{i_1} \dots X_n^{i_n}.$$

Since for every $h, h' \in N_{\overline{\mathcal{G}}}(P)$, $h^{-1}h'h \in P$ if and only if $h' \in P$, by equation (100), we have

$$\text{Res}_P^{N_{\overline{\mathcal{G}}}(P)}(\kappa_{N_{\overline{\mathcal{G}}}(P)}(h')) = \begin{cases} \sum_{x \in W_{\overline{\mathcal{G}}}(P)} \kappa_P((x^{\tau})^{-1} h'^{\tau} x^{\tau}), & \text{if } h' \in P, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that each term $c_{g, (i_1, \dots, i_n)} \text{Res}_P^{N_{\overline{\mathcal{G}}}(P)} \left(\text{Res}_{N_{\overline{\mathcal{G}}}(P)}^{\overline{\mathcal{G}}}(\kappa(g)) \right) X_1^{i_1} \dots X_n^{i_n} \in \mathcal{T}_P$. \square

Proof of Theorem 7.40(iv): Note that $p\mathcal{T}_P \subset p\mathcal{R}[\overline{\mathcal{G}}]^\tau$. Taking $I = p\mathcal{R}[\overline{\mathcal{G}}]^\tau$ in Lemma 7.14, we have an isomorphism $I \xrightarrow{\log_{\overline{\mathcal{G}}}^{-1}} 1 + I$. Then it follows from Proposition 7.17 and Lemma 7.41, that the congruences follow if $\eta_P \circ \text{Res}_P^{\overline{\mathcal{G}}}(\mathcal{R}[\text{Conj}(\overline{\mathcal{G}})]^\tau) \subset \mathcal{T}_P$. Since η_P preserves \mathcal{T}_P , it follows from the Lemma 7.42 above, that the containment holds and hence the congruence. \square

This finishes the proof of Theorem 7.40, and the Theorem D. By this theorem, to show that an element is in the image of $K'_1(\mathbb{I}[[\mathcal{G}]])$ under the map $\Theta^{\mathcal{G}}$ it is sufficient to verify the statements in Theorem 7.40.

Definition 7.43. We now consider the map $\mathcal{L} = (\mathcal{L}_P) : \Phi^{\overline{\mathcal{G}}} \rightarrow \Psi^{\overline{\mathcal{G}}}$, defined by

$$\mathcal{L}_P((x_C)) = \begin{cases} \frac{1}{p^2|P|} \log \left(\frac{\alpha_P(\theta_P^{\overline{\mathcal{G}}}(x_P))^{p|P|}}{u_P^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x_C)))} \right), & \text{if } P \notin C(\overline{\mathcal{G}}) \\ \frac{1}{p} \log \left(\frac{\alpha_P(\theta_P^{\overline{\mathcal{G}}}(x_P))^{p|P|}}{u_P^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x_C)))} \right), & \text{if } P \in C(\overline{\mathcal{G}}), P \neq \{1\} \\ \frac{1}{p} \log \left(\frac{\alpha_{\{1\}}((x_{\{1\}}))^p}{\varphi((x_{\{1\}})(u_{\{1\}}^{\overline{\mathcal{G}}}(\alpha(\theta^{\overline{\mathcal{G}}}(x_{\{1\}}))))} \right), & \text{if } P = \{1\}. \end{cases}$$

Lemma 7.44. The following sequence is exact

$$1 \longrightarrow \mu(\mathcal{O}) \times \mathbb{W} \times \mathcal{G}^{ab} \longrightarrow \Phi^{\overline{\mathcal{G}}} \xrightarrow{\mathcal{L}} \Psi^{\overline{\mathcal{G}}} \xrightarrow{\omega} \mathbb{W} \times \mathcal{G}^{ab} \longrightarrow 1.$$

More precisely, the map $\mu(\mathcal{O}) \times \mathbb{W} \times \mathcal{G}^{ab} \rightarrow \Phi^{\overline{\mathcal{G}}} \subset \prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})^\times$ is the composition

$$\mu(\mathcal{O}) \times \mathbb{W} \times \mathcal{G}^{ab} \longrightarrow \prod_{P \leq \overline{\mathcal{G}}} \mu(\mathcal{O}) \times \mathbb{W} \times U_P^{ab} \hookrightarrow \prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})^\times$$

where the first map is the identity on $\mu(\mathcal{O})$ and the transfer homomorphism from \mathcal{G}^{ab} to U_P^{ab} for each $P \leq \overline{\mathcal{G}}$.

Proof. Clearly the image of \mathcal{L} is contained in $\prod_{P \leq \overline{\mathcal{G}}} \mathbb{Q}_p \otimes \mathbb{I}(U_P^{ab})$. To show that the image is contained in $\Psi^{\overline{\mathcal{G}}} \prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})$, we show that the conditions defining the set $\Psi^{\overline{\mathcal{G}}}$ are satisfied. Below we show how the first condition defining $\Psi^{\overline{\mathcal{G}}}$ can be shown. The rest of the conditions can be demonstrated easily from the conditions defining $\Phi^{\mathcal{G}}$ ([Kak13, Lemma 88]).

Let $P \leq P' \leq \overline{\mathcal{G}}$ such that $[P', P] \leq P$ with P a non-trivial cyclic group if $[P', P'] \neq P$. We then have three cases to consider: (i) P is not cyclic, (ii) P is cyclic but P' is not cyclic, and (iii) P' is cyclic.

Case (i): Let P be not cyclic. Letting C' run through all cyclic subgroups of $\overline{\mathcal{G}}$ with $C'^p \leq P'$ and C run through all cyclic subgroups of $\overline{\mathcal{G}}$ with $C^p \leq P$, we have

$$\begin{aligned} \text{Tr}_P^{P'}(\mathcal{L}_{P'}((x_C))) &= \text{Tr}_P^{P'} \left(\frac{1}{p^2|P'|} \log \left(\frac{\alpha_{P'}(x_{P'})^{p|P'|}}{u_{P'}^{\overline{\mathcal{G}}}(\alpha((x_C)))} \right) \right) \\ &= \text{Tr}_P^{P'} \left(\frac{1}{p^2|P'|} \log \left(\frac{(x_{P'})^{p^2|P'|}}{\prod_{C'} \varphi(\alpha_{C'}(x_{C'}))^{|C'|}} \right) \right) \\ &= \frac{1}{p^2|P'|} \log \left(\frac{Nr_P^{P'}(x_{P'})^{p^2|P'|}}{Nr_P^{P'}(\prod_{C'} \varphi(\alpha_{C'}(x_{C'}))^{|C'|})} \right) \\ &= \frac{1}{p^2|P'|} \log \left(\frac{\Pi_P^{P'}(x_{P'})^{p^2|P'|}}{\prod_C \varphi(\alpha_C(x_C))^{p|C|}} \right), \text{ by first condition of } \Phi^{\mathcal{G}} \\ &= \Pi_P^{P'} \left(\frac{1}{p^2|P'|} \log \left(\frac{\alpha_P(x_P)^{p|P'|}}{\prod_C \varphi(\alpha_C(x_C))^{|C|}} \right) \right) \\ &= \Pi_P^{P'}(\mathcal{L}_P((x_C))). \end{aligned}$$

Case (ii): Let P be cyclic but P' be not cyclic. Let C run through all cyclic subgroups of $\overline{\mathcal{G}}$ with $C^p \leq P$. Then

$$\eta_P(\mathrm{Tr}_P^{P'}(\mathcal{L}_{P'}((x_C)))) = \Pi_P^{P'} \left(\eta_P \left(\frac{1}{p^2|P|} \log \left(\frac{(x_P)^{p^2|P|}}{\prod_C \varphi(\alpha_C(x_C))^{|C|}} \right) \right) \right).$$

Since $\alpha_P(\varphi(\alpha_C(x_C))) = \alpha_P(\alpha_C(x_C))^p$ (resp. 1) if $C^p = P$ (resp. $C^p \neq P$), letting C run through all cyclic subgroups of $\overline{\mathcal{G}}$ with $C^p = P$, we have

$$\begin{aligned} \eta_P(\mathrm{Tr}_P^{P'}(\mathcal{L}_{P'}((x_C)))) &= \Pi_P^{P'} \left(\frac{1}{p} \log \left(\frac{\alpha(x_P)}{\prod_C \varphi(\alpha_C(x_C))} \right) \right) \\ &= \Pi_P^{P'}(\mathcal{L}_P((x_C))). \end{aligned}$$

Case (iii): Let P' be cyclic. This case follows from the following lemma:

Lemma 7.45. *Let $P \leq P' \leq \overline{\mathcal{G}}$ such that $[P' : P] = p$. Let $C \in C(\overline{\mathcal{G}})$ be such that C^p is contained in P' but not in P . Then $\mathrm{Nr}_P^{P'}(\varphi(\alpha_C(x_C))) = 1$ in $\mathbb{I}(U_P/[U_{P'}, U_{P'}])$.*

Proof. By definition, we have $\alpha_C(x_C) = \frac{x_C^p}{\prod_{k=0}^{p-1} \omega_C^k(x_C)}$. Therefore, $\varphi(\alpha_C(x_C)) = \frac{\varphi(x_C^p)}{\prod_{k=0}^{p-1} \varphi(\omega_C^k(x_C))} = \frac{\varphi(x_C^p)}{\prod_{k=0}^{p-1} \omega_{C^p}^k(\varphi(x_C))}$. Since $\mathrm{Nr}_P^{P'}(\varphi(\alpha_C(x_C))) = \prod_{k=0}^{p-1} \omega_{C^p}^k(\varphi(x_C))$ by a straightforward generalization of [Kak13, Lemma 50], the lemma follows. \square

Regarding the second and third conditions defining $\Psi^{\overline{\mathcal{G}}}$, both follow easily from (C2) and (C4) respectively.

Finally, to show that the image of \mathcal{L} is contained in $\prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})$, it is enough to note that $\mathcal{L}_P((x_C)) \in \mathcal{T}_P$ for all $P \in C(\overline{\mathcal{G}})$. Then by Proposition 7.11 it follows that $\mathrm{im}(\mathcal{L}) \subseteq \prod_{P \leq \overline{\mathcal{G}}} \mathbb{I}(U_P^{ab})$.

The exactness of the four term sequence can also shown as in [Kak13, Lemma 88]. However, there is one crucial input which is the fact that the only torsion elements of $\mathbb{I}(U_P^{ab})^\times$ are contained in $\mu(\mathcal{O}) \times \mathbb{W} \times U_P^{ab}$, which is a generalization of a Theorem of Higman [Hig40]. This input is provided by Proposition 7.23 and Corollary 7.22. We then have the following commutative diagram:

$$\begin{array}{ccc} K'_1(\mathbb{I}[[\mathcal{G}]]) & \xrightarrow{\xi} & \mathbb{I}(Z)[\mathrm{Conj}(\overline{\mathcal{G}})]^\tau \\ \Theta^{\overline{\mathcal{G}}} \downarrow & & \downarrow \beta^{\overline{\mathcal{G}}} \\ \Phi^{\overline{\mathcal{G}}} & \xrightarrow{\mathcal{L}} & \Psi^{\overline{\mathcal{G}}}. \end{array}$$

In other words, the image of $\Theta^{\overline{\mathcal{G}}}$ is contained in $\Phi^{\overline{\mathcal{G}}}$. \square

In the same way, we can prove that the image of $\widehat{\Theta_{\mathcal{J}}^{\overline{\mathcal{G}}}}$ is contained in $\widehat{\Phi_{\mathcal{J}}^{\overline{\mathcal{G}}}}$, which we record below.

Theorem 7.46. *The image of $\widehat{\Theta_{\mathcal{J}}^{\overline{\mathcal{G}}}}$ under the logarithm map is contained in $\widehat{\Phi_{\mathcal{J}}^{\overline{\mathcal{G}}}}$.*

Theorem 7.47. *The map $\Theta^{\overline{\mathcal{G}}} : K'_1(\mathbb{I}[[\mathcal{G}]]) \rightarrow \Phi^{\overline{\mathcal{G}}}$ is an isomorphism.*

Proof. The lemmas regarding the restrictions under integral logarithms give us the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu(\mathcal{O}) \times \mathbb{W} \times \mathcal{G}^{ab} & \longrightarrow & K'_1(\mathbb{I}[[\mathcal{G}]]) & \xrightarrow{\xi} & \mathbb{I}(Z)[\mathrm{Conj}(\overline{\mathcal{G}})]^\tau \longrightarrow \mathbb{W} \times \mathcal{G}^{ab} \longrightarrow 1 \\ & & \downarrow & & \downarrow \Theta^{\overline{\mathcal{G}}} & \cong \downarrow \beta^{\overline{\mathcal{G}}} & \downarrow \\ 1 & \longrightarrow & \mu(\mathcal{O}) \times \mathbb{W} \times \mathcal{G}^{ab} & \longrightarrow & \Phi^{\overline{\mathcal{G}}} & \xrightarrow{\mathcal{L}} & \Psi^{\overline{\mathcal{G}}} \longrightarrow \mathbb{W} \times \mathcal{G}^{ab} \longrightarrow 1. \end{array}$$

The Five Lemma then gives the result. \square

Theorem 7.48. *The map $\Theta_{\mathcal{G}}^{\bar{\mathcal{G}}}$ maps $K'_1(\mathbb{I}[[\mathcal{G}]])_{\mathcal{G}}$ into $\Phi_{\mathcal{G}}^{\bar{\mathcal{G}}}$. Further*

$$\Phi_{\mathcal{G}}^{\bar{\mathcal{G}}} \cap \prod_{P \leq \bar{\mathcal{G}}} \mathbb{I}(U_P^{ab})^{\times} = \text{im}(\Theta^{\bar{\mathcal{G}}}).$$

Proof. Note that $\widehat{\Phi_{\mathcal{G}}^{\bar{\mathcal{G}}}} \cap \prod_{P \leq \bar{\mathcal{G}}} \mathbb{I}(U_P)^{\times}_{\mathcal{G}} = \Phi_{\mathcal{G}}^{\bar{\mathcal{G}}}$. By Theorem 7.40 and 7.46, it follows that

$$\text{im}(\Theta_{\mathcal{G}}^{\bar{\mathcal{G}}}) \subset \Phi_{\mathcal{G}}^{\bar{\mathcal{G}}}$$

Further, since $\Phi_{\mathcal{G}}^{\bar{\mathcal{G}}} \cap \prod_{P \leq \bar{\mathcal{G}}} \mathbb{I}(U_P^{ab})^{\times} = \Phi_{\mathcal{G}}^{\bar{\mathcal{G}}}$, we get from Theorem 7.40, that

$$\Phi_{\mathcal{G}}^{\bar{\mathcal{G}}} \cap \prod_{P \leq \bar{\mathcal{G}}} \mathbb{I}(U_P^{ab})^{\times} = \text{im}(\Theta^{\bar{\mathcal{G}}}).$$

□

8 Relations between the congruences over $\mathbb{I}[[\mathcal{G}]]$ and $\mathbb{Z}_p[[\mathcal{G}]]$

8.1 Congruences over $\mathbb{Z}_p[[\mathcal{G}]]$

We first recall the main result of Kakde [Kak13]. As in the previous section, we fix a lift $\tilde{\Gamma}$ of Γ in \mathcal{G} . Then we can identify \mathcal{G} with $H \rtimes \Gamma$. Fix $e \in \mathbb{N}$ such that $\tilde{\Gamma}^{p^e} \subset Z(\mathcal{G})$, and put $\bar{\mathcal{G}} := \mathcal{G}/\Gamma^{p^e}$ and $R := \Lambda_{\mathcal{O}}(\tilde{\Gamma}^{p^e})$. Then $\Lambda_{\mathcal{O}}(G) \cong R[\bar{\mathcal{G}}]^{\tau}$, the twisted group ring with multiplication

$$(h\tilde{\gamma}^a)^{\tau}(h'\tilde{\gamma}^b)^{\tau} = \tilde{\gamma}^{p^e[\frac{a+b}{b}]}(h\tilde{\gamma}^a.h'\tilde{\gamma}^b)^{\tau},$$

where g^{τ} is the image of $g \in \mathcal{G}$ in $R[\bar{\mathcal{G}}]^{\tau}$.

Let P be a subgroup of $\bar{\mathcal{G}}$ and U_P be the inverse image of P in \mathcal{G} . Recall that, $N_{\bar{\mathcal{G}}}P :=$ the normalizer of P in $\bar{\mathcal{G}}$, $W_{\bar{\mathcal{G}}}(P) := N_{\bar{\mathcal{G}}}P/P$, and $C(\bar{\mathcal{G}}) :=$ set of cyclic subgroups of $\bar{\mathcal{G}}$. If $P \in C(\bar{\mathcal{G}})$, then U_P is a rank one abelian subquotient of \mathcal{G} , and for every $P \in C(\bar{\mathcal{G}})$, set

$$T_P := \left\{ \sum_{g \in W_{\bar{\mathcal{G}}}(P)} g^{\tau} x (g^{\tau})^{-1} \mid x \in R[P]^{\tau} \right\}.$$

Let $P \leq P' \leq \bar{\mathcal{G}}$. Then consider the homomorphism $\mathbb{Z}_p[[\mathcal{G}]] \rightarrow \mathbb{Z}_p[[\mathcal{G}]]$ given by $x \mapsto \sum_{g \in P'/P} \tilde{g}x\tilde{g}^{-1}$, where \tilde{g} is a lift of g . We define $T_{P,P'}$ to be the image of this homomorphism. For two subgroups P, P' of $\bar{\mathcal{G}}$ with $[P', P'] \leq P \leq P'$ consider

$$(111) \quad \begin{aligned} \text{nr}_P^{P'} : \Lambda_{\mathcal{O}}(U_{P'}^{ab})^{\times} &\rightarrow \Lambda_{\mathcal{O}}(U_P/[U_{P'}, U_{P'}])^{\times}, \quad (\text{the norm map}), \\ \pi_P^{P'} : \Lambda_{\mathcal{O}}(U_{P'}^{ab}) &\rightarrow \Lambda_{\mathcal{O}}(U_P/[U_{P'}, U_{P'}]), \quad (\text{the projection map}). \end{aligned}$$

For $P \in C(\bar{\mathcal{G}})$ with $P \neq (1)$, fix a homomorphism $\omega_P : P \rightarrow \bar{\mathbb{Q}}_p^{\times}$ of order p , and also a homomorphism $\omega_1 := \omega_{\{1\}} : \tilde{\Gamma}^{p^e} \rightarrow \bar{\mathbb{Q}}_p^{\times}$ of order p . The homomorphism ω_P induce the following homomorphism which we again denote by the same symbol:

$$(112) \quad \omega_P : \Lambda_{\mathcal{O}}(U_P)^{\times} \rightarrow \Lambda_{\mathcal{O}}(U_P)^{\times}, g \mapsto \omega_P(g)g.$$

For $P \leq \bar{\mathcal{G}}$, consider the homomorphism $\alpha_P : \Lambda_{\mathcal{O}}(U_P)_{\mathcal{S}}^{\times} \rightarrow \Lambda_{\mathcal{O}}(U_P)_{\mathcal{S}}^{\times}$ defined by

$$(113) \quad \alpha_P(x) := \begin{cases} x^p \varphi(x)^{-1} & \text{if } P = \{1\} \\ x^p (\prod_{k=0}^{p-1} \omega_P^k(x))^{-1} & \text{if } P \neq \{1\} \text{ and cyclic} \\ x^p & \text{if } P \text{ is not cyclic.} \end{cases}$$

Note that, for all $P \leq \overline{\mathcal{G}}$, there is an action of \mathcal{G} and $\overline{\mathcal{G}}$ on U_P^{ab} by conjugation since $\tilde{\Gamma}^{p^e}$ is central. Now consider the following map

$$(114) \quad K'_1(\mathbb{Z}_p[[\mathcal{G}]]_S) \longrightarrow K'_1(\mathbb{Z}_p[[U]]_S) \longrightarrow K'_1(\mathbb{Z}_p[[U^{ab}]]_S) \longrightarrow \mathbb{Z}_p[[U^{ab}]]_S^\times \subset Q(\mathbb{Z}_p[[U^{ab}]]_S)^\times.$$

Taking all the U^{ab} in $\Sigma(\mathcal{G})$ we get the following homomorphism

$$(115) \quad \theta_{\Sigma(\mathcal{G})} : K'_1(\mathbb{Z}_p[[\mathcal{G}]]_S) \longrightarrow \prod_{U^{ab} \in \Sigma(\mathcal{G})} Q(\mathbb{Z}_p[[U^{ab}]]_S)^\times.$$

For any subgroup P of $\overline{\mathcal{G}}$, we write $\theta_{\mathcal{G}, ab}^P$ for the following natural composite homomorphism

$$K'_1(\mathbb{Z}_p[[\mathcal{G}]]) \xrightarrow{\theta_{\mathcal{G}}^P} K_1(\mathbb{Z}_p[[U_P]]) \longrightarrow K_1(\mathbb{Z}_p[[U_P^{ab}]]) \cong \mathbb{Z}_p[[U_P^{ab}]]^\times,$$

where the isomorphism is induced by taking determinants over $\mathbb{Z}_p[[U_P^{ab}]]$.

Proposition 8.1. [Kak13] *Let \mathcal{G} be a rank one pro- p group. Then the set $\Sigma(\mathcal{G}) := \{U_P^{ab} : P \leq \overline{\mathcal{G}}\}$ satisfies the condition $(*)$. Further, an element $(\xi_{\mathcal{A}})_{\mathcal{A}} \in \prod_{\mathcal{A} \in \Sigma(\mathcal{G})} \Lambda_{\mathcal{O}}(\mathcal{A})^\times$ belongs to $\text{im}(\theta_{\Sigma(\mathcal{G})})$ if and only if it satisfies all of the following three conditions.*

(i) *For all subgroups P, P' of $\overline{\mathcal{G}}$ with $[P', P'] \leq P \leq P'$, one has*

$$(116) \quad \text{nr}_P^{P'}(\xi_{U_P^{ab}}) = \pi_P^{P'}(\xi_{U_{P'}^{ab}}).$$

(ii) *For all subgroups P of $\overline{\mathcal{G}}$ and all g in $\overline{\mathcal{G}}$ one has $\xi_{gU_P^{ab}g^{-1}} = g\xi_{U_P^{ab}}g^{-1}$.*

(iii) *For every $P \in \overline{\mathcal{G}}$ and $P \neq (1)$, we have*

$$\text{ver}_P^{P'}(\xi_{U_{P'}^{ab}}) \equiv \xi_{U_P^{ab}} \pmod{T_{P, P'}}.$$

(iv) *For all $P \in C(\overline{\mathcal{G}})$ one has $\alpha_P(\xi_{U_P^{ab}}) \equiv \prod_{P' \in C_P(\overline{\mathcal{G}})} \alpha_{P'}(\xi_{U_{P'}^{ab}}) \pmod{pT_P}$.*

We recall from [Kak13] the following set of congruences $\phi^{\overline{\mathcal{G}}}$, which are defined as follows.

Definition 8.2. Let $\phi^{\overline{\mathcal{G}}}$ denote the subgroup of $\prod_{P \leq \overline{\mathcal{G}}} \Lambda_{\mathcal{O}}(U_P^{ab})^\times$ consisting of tuples (x_P) satisfying the conditions of the above theorem.

8.2 Relation between the congruences

Consider the following commutative diagram, which is easily seen to be induced by each specialization map:

$$(117) \quad \begin{array}{ccc} K_1(\mathbb{I}[[\mathcal{G}]]) & \longrightarrow & \Phi_{\mathcal{H}}^{\overline{\mathcal{G}}} \\ \downarrow & & \downarrow \\ K_1(\mathbb{Z}_p[[\mathcal{G}]]) & \longrightarrow & \phi^{\overline{\mathcal{G}}}. \end{array}$$

From this commutative diagram it is easy to see that the congruences over $\mathbb{I}[[\mathcal{G}]]$ implies the congruences over $\mathbb{Z}_p[[\mathcal{G}]]$. The following proposition easily follows from this commutative diagram and the interpolation formula of the p -adic L-function over $\mathbb{I}[[\mathcal{G}]]$.

Proposition 8.3. *Let $\Xi \in K_1(\mathbb{I}[[\mathcal{G}]])$ be a p -adic L-function over $\mathbb{I}[[\mathcal{G}]]$, then under every specialization map ϕ_k , $\phi_k(\Xi) \in K_1(\mathbb{Z}_p[[\mathcal{G}]])$ is a p -adic L-function over $\mathbb{Z}_p[[\mathcal{G}]]$.*

9 Application to p -adic L-function

In this section, we generalize some of the results of Ritter-Weiss that has been used to prove the congruences and hence the main conjecture in some important cases. More precisely, we generalize the torsion congruences which have been used as a basic step in proving the congruences.

9.1 Torsion Congruences and p -adic L-function

Let $F^{ab,p}$ be the maximal pro- p abelian extension of F that is unramified outside p and ∞ . We set $\text{Gal}_F = \text{Gal}(F^{ab,p}/F)$. Let $\{f_\kappa\}$ be a family of Hilbert modular forms over F which is parametrized by an irreducible component \mathbb{I} of the universal cyclotomic deformation ring \mathcal{R}_F . In fact, \mathbb{I} is a finite flat algebra over $\mathbb{Z}_p[[\mathbb{W}]]$, where \mathbb{W} is the torsion free part of \mathcal{O}_p as in Section 2.2. As in Definition 2.1, let $\phi_\kappa : \mathbb{I} \rightarrow \mathbb{Z}_p$ denote an arithmetic point of weight κ . Set $\mathcal{P}_\kappa = \ker(\phi_\kappa)$. This induces an algebra homomorphism $\mathbb{I}[[G_F]] \rightarrow \mathbb{Z}_p[[G_F]]$ which we again denote by ϕ_κ . Furthermore, we extend ϕ_κ to $G_F \times \mathbb{W}$ by setting $\phi_\kappa(g) = 1$ for all $g \in G_F$. The extended character is still denoted by ϕ_κ . Further note that $\mathfrak{m} = \langle p, \mathcal{P}_\kappa \rangle$ is the maximal ideal of \mathbb{I} , and we have a natural homomorphism $\mathbb{I}/\mathfrak{m}[[G_F]] \rightarrow \mathbb{F}_p[[G_F]]$.

For any integer $m \geq 0$, let $\psi_m : G_F \rightarrow \mathbb{Z}_p^\times$ be a character of the form $\psi \chi_p^m$ where χ_p is the p -adic cyclotomic character, and ψ is a character of finite order. We again extend the character ψ_m to the group $G_F \times \mathbb{W}$ by setting ψ_m to be equal to 1 on \mathbb{W} .

Suppose that there exists a measure μ_F in $\mathbb{I}[[G_F]]$ that interpolates the critical values of each of the representations $\text{Ad}^0(\rho_{f_\kappa}) \otimes \psi$ for characters ψ of G_F . More precisely,

$$\int_{G_F \times \mathbb{W}} \chi(g) \phi_\kappa(g) d\mu_F(g) = L^*(\text{Ad}^0(\rho_{f_\kappa}) \otimes \chi, 0),$$

where $L^*(\text{Ad}^0(\rho_{f_\kappa}) \otimes \chi, 0)$ involves the critical value $L(\text{Ad}^0(\rho_{f_\kappa}) \otimes \chi, 0)$ twisted by the finite order character χ , some archimedean periods related to $\text{Ad}^0(\rho_{f_\kappa})$, and some Euler factors removed, as in the interpolation formula in (80).

Now, we consider the measure defined by

$$\mu_\kappa(g) = \int_{\gamma' \in \{g\} \times \mathbb{W}} \phi_\kappa(\gamma') d\mu_F(\gamma'), \text{ for all } g \in G_F.$$

Then $\mu_\kappa \in \mathbb{Z}_p[[G_F]]$, and

$$(118) \quad \int_{G_F} \psi_m(g) d\mu_\kappa(g) = \int_{G_F} \psi_m(g) \int_{\gamma' \in \{g\} \times \mathbb{W}} \phi_\kappa(\gamma') d\mu_F(\gamma') = \int_{G_F \times \mathbb{W}} \psi_m(x) \phi_\kappa(x) d\mu_F(x).$$

Therefore, $\int_{G_F} \chi(g) d\mu_\kappa(g) = L^*(\text{Ad}^0(\rho_{f_\kappa}) \otimes \chi, 1)$ for all finite order characters χ . It follows that $\mu_\kappa \in \mathbb{Z}_p[[G_F]]$ is a p -adic L-function interpolating the special values of $\text{Ad}^0(f_\kappa)$ twisted by all finite order characters of G_F .

Let $\delta^{(x)}$ be the characteristic function of a coset of an open subgroup U . Then $\delta^{(x)}(g) = \sum_j c_j \chi_j(g)$, for some $c_j \in \mathbb{Z}_p$. Let

$$(119) \quad L^*(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi) = e_p(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi) \mathcal{L}_p(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi) \frac{L_{(S,p)}(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi, 0)}{\Omega_\infty(\text{Ad}^0(\tilde{f}_\kappa)/F')},$$

where $\mathcal{L}_p(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi) := \prod_{p|p} \mathcal{L}_p(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi)$ comes from the Euler factors at primes lying above p , and $e_p(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi)$ is the product of the local epsilon factors above p . We define,

$$L^*(\text{Ad}^0(f_\kappa), \delta^{(x)}) = \sum_j L^*(\text{Ad}^0(f_\kappa), \chi_j, 0).$$

Consider the cyclotomic character $\mathcal{N}_F : G_F \rightarrow \mathbb{Z}_p$. Then an open subgroup U of G_F is said to be admissible if $\mathcal{N}_F(U) \subset 1 + p\mathbb{Z}_p$, and define $m_F(U) \geq 1$, by $\mathcal{N}_F(U) = 1 + p^{m_F(U)}\mathbb{Z}_p$. Then we

have the following lemma which is a generalization of a result in [RW08]. The proof presented is also a straight forward generalization of the proof due to Ritter and Weiss in loc. cit.

Lemma 9.1. $\mathbb{I}[[G_F]]$ is the inverse limit of the system $\mathbb{I}[G_F/U]/p^{m_F(U)}\mathbb{I}[G_F/U]$, with U running over the cofinal system of admissible open subgroups of G_F .

Proof. If V is an admissible open subgroup of $G_{F'}$ and U is an admissible open subgroup of G_F in $\text{ver}^{-1}(V)$, then $m_F(U) \geq m_{F'}(V) - 1$. Consider the natural map

$$\mathbb{I}[[G]] \longrightarrow \varprojlim_U \mathbb{I}[G_F/U]/p^{m_F(U)}\mathbb{I}[G_F/U].$$

We first show that this map is surjective. Let $(x_U)_U \in \varprojlim_U \mathbb{I}[G_F/U]/p^{m_F(U)}\mathbb{I}[G_F/U]$. Then for any $V \subseteq U$, consider the map

$$\mathbb{I}[G_F/V]/p^{m_F(V)}\mathbb{I}[G_F/V] \longrightarrow \mathbb{I}[G_F/U]/p^{m_F(V)}\mathbb{I}[G_F/U]$$

and let the image of x_V be denoted by $\overline{x_V}$. Note that fixing U and taking the projective limit over $m_F(V)$, we have

$$(120) \quad \mathbb{I}[G_F/U] \cong \varprojlim_V \mathbb{I}[G_F/U]/p^{m_F(V)}\mathbb{I}[G_F/U].$$

Indeed, we have $\mathbb{I}[G_F/U]/p^k\mathbb{I}[G_F/U] \cong (\mathbb{I}/p^k)[G_F/U]$, for every nonnegative integer k . Taking projective limit with respect to k , we have $\mathbb{I}[G_F/U] = \varprojlim_k \mathbb{I}[G_F/U]/p^k\mathbb{I}[G_F/U]$. The lemma follows. \square

Lemma 9.2. The image of the measure $\mu_F \in \mathbb{I}[[G_F]]$ in $\mathbb{I}[G_F/U]/\mathfrak{m}^k$ is given by $\sum_{\bar{g} \in G/U} \mu_F(\cdot, \delta^{\bar{g}})\bar{g} \pmod{\mathfrak{m}^k}$.

Proof. Consider the measure $\mu_F \in \mathbb{I}[[G]]$. Then μ_F is a measure on $\mathbb{W} \times G$. Then, it is a standard fact that the image of μ_F in $\mathbb{I}[G/U]$ is given by $\sum_{\bar{g} \in G/U} \mu_F(\cdot, \delta^{\bar{g}})\bar{g}$, where $\mu(\cdot, \delta^{\bar{g}}) \in \mathbb{I}$. \square

Now let F' be a totally real extension of F contained in $F^{ab,p}$. Consider the base change Hilbert modular form over F' . Let \tilde{f}_κ denote the base-change of f_κ to F' , and let \mathbb{J} be the irreducible component to which \tilde{f}_κ belongs. Let $\mu_{F'}$ be the measure in $\mathbb{J}[[G_{F'}]]$ interpolating the special values of $\text{Ad}^0(\tilde{f}_\kappa)$.

Lemma 9.3. Let y be a coset of a Δ -stable admissible open subgroup of $G_{F'}$, where $\Delta = G(F'/F)$. Then

$$(121) \quad L^*(\text{Ad}^0(\tilde{f}_\kappa)/F', \delta_F^{(y)}) = L^*(\text{Ad}^0(\tilde{f}_\kappa)/F', \delta_{F'}^{(y^\gamma)}),$$

for all $\gamma \in \Delta$. Further, let $\tilde{\mu}_{F'}$ be the image of $\mu_{F'}$ under the map $\mathbb{J}[[G_{F'}]] \longrightarrow \mathbb{I}[[G_{F'}]]$. Then $\tilde{\mu}_{F'} \in \mathbb{I}[[G_{F'}]]^\Delta$.

Proof. Here $\gamma \in \Delta$ acts on $G_{F'}$ by conjugation and trivially on \mathbb{I} . It is enough to prove for finite order characters χ of $G_{F'}$. Recall that

$$(122) \quad L^*(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi) = e_p(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi) \mathcal{L}_p(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi) \frac{L_{(S,p)}(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi, 0)}{\Omega_\infty(\text{Ad}^0(\tilde{f}_\kappa)/F')},$$

where $L_{(S,p)}(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi, 0)$ is the critical value at $s = 0$ of the L -function $L(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi, s)$ with the Euler factors at S and those above p removed.

Note that by induction of L -functions, we have,

$$(123) \quad \begin{aligned} L_{(S,p)}(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi, s) &= L_{(S,p)}(\text{Ad}^0(\tilde{f}_\kappa)/F, \text{ind}_F^{F'} \chi, s) \\ &= L_{(S,p)}(\text{Ad}^0(\tilde{f}_\kappa)/F, \text{ind}_F^{F'} \chi^\gamma, s) \\ &= L_{(S,p)}(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi^\gamma, s). \end{aligned}$$

We also have $\mathcal{L}_p(\text{Ad}^0(\tilde{f}_\kappa)/F, \chi) = \prod_{\mathfrak{p}|p} \mathcal{L}_{\mathfrak{p}}(\text{Ad}^0(\tilde{f}_\kappa)/F', \chi)$ and therefore the equality in the lemma holds.

Let κ' be the weight of \tilde{f}_κ and $\phi_{\kappa'}$ be any arithmetic specialization of weight κ' . Now

$$\begin{aligned}
(\mu_{F'})^\gamma(\phi_{\kappa'}, \delta_{F'}^{(y)}) &= \mu_{F'}(\phi_{\kappa'}, \delta_{F'}^{(y^\gamma)}) \\
&= L^*(\text{Ad}^0(\tilde{f}_{\kappa'})/F', \delta_{F'}^{(y^\gamma)}) \\
&= L^*(\text{Ad}^0(\tilde{f}_\kappa)/F', \delta_{F'}^{(y)}) \\
&= \mu_{F'}(\phi_{\kappa'}, \delta_{F'}^{(y)}).
\end{aligned}
\tag{124}$$

In fact, we have $(\mu_{F'})^\gamma(\phi_{\kappa'}, \chi) = \mu_{F'}(\phi_{\kappa'}, \chi)$, for any finite order χ of $G_{F'}$. Since this holds for all the arithmetic specializations $\phi_{\kappa'}$, the measures $(\mu_{F'})^\gamma(\cdot, \chi) = \mu_{F'}(\cdot, \chi)$. Indeed, the measures $(\mu_{F'})^\gamma(\cdot, \chi)$ and $\mu_{F'}(\cdot, \chi)$ on \mathbb{W} are equal at infinitely many characters, they are equal. This further implies that $(\mu_{F'})^\gamma = \mu_{F'}$, for all $\gamma \in \Delta$. Since the morphism $\mathbb{J} \rightarrow \mathbb{I}$ is equivariant with respect to Δ , $(\tilde{\mu}_{F'})^\gamma = \tilde{\mu}_{F'}$, for all $\gamma \in \Delta$. Therefore $\tilde{\mu}_{F'} \in \mathbb{I}[[G_{F'}]]^\Delta$. \square

Theorem 9.4. *Let $\mu_F \in \mathbb{I}[[G_F]]$ be a measure interpolating all the critical values of each arithmetic specialization twisted by finite order characters of G_F . Similarly, let $\mu_{F'} \in \mathbb{I}[[G_{F'}]]$ interpolating the critical values of the base change of each arithmetic specialization. Recall the trace ideal $\mathcal{T} \in \mathbb{I}[[G_{F'}]]^\Delta$ generated by the elements $\Sigma_{\gamma \in \Delta} \alpha^\gamma$, with $\alpha \in \mathbb{I}[[G_{F'}]]$. Then the congruence*

$$\text{ver}(\mu_F) \equiv \tilde{\mu}_{F'} \pmod{\mathcal{T}}
\tag{125}$$

hold if and only if for every locally constant \mathbb{Z}_p -valued function ϵ of $G_{F'}$ satisfying $\epsilon^\gamma = \epsilon$ for all $\gamma \in \Delta$ we have the following congruences

$$\int_{G_F} \epsilon \circ \text{ver}(x) d\mu_F(x) \equiv \int_{G_{F'}} \epsilon(x) d\tilde{\mu}_{F'}(x) \pmod{p\mathbb{I}}.
\tag{126}$$

Proof. The necessary part is clear and we need only prove the sufficient part. Consider the components of the images of $\tilde{\mu}_{F'}$ and $\text{ver}(\mu_F)$ in $\mathbb{I}[G_{F'}/V]/p^{m_{F'}(V)-1}$ for a Δ -stable admissible open subgroup V of $G_{F'}$. We denote the component obtained by evaluating $\tilde{\mu}_{F'}$ at $\delta_{F'}^{(y)}$ by $\tilde{\mu}_{F'}(\cdot, \delta_{F'}^{(y)})$ and the component obtained by evaluating μ_F at $\delta_{F'}^{(x)}$ by $\tilde{\mu}_F(\cdot, \delta_{F'}^{(x)})$. Let $U := \text{ver}^{-1}(V) \subseteq G_F$, then $\text{ver}(\mu_F)$ is the image under the transfer map of the U -component of μ_F . These components are the images of

- (i) $\sum_{y \in G_{F'}/V} \tilde{\mu}_{F'}(\cdot, \delta_{F'}^{(y)}) y,$
- (ii) $\sum_{x \in G_F/U} \mu_F(\cdot, \delta_{F'}^{(x)}) \text{ver}(x)$

in $(\mathbb{I}[G_{F'}/V]/p^{m_{F'}(V)-1})^\Delta$. Let $\mathcal{T}(V)$ be the image of the trace ideal \mathcal{T} in $(\mathbb{I}[G_{F'}/V]/p^{m_{F'}(V)-1})^\Delta$. We consider the following two cases:

Case(i): y is fixed by Δ . In this case, $\delta_{F'}^{(y)}$ is a locally constant function as in the statement of the theorem. Now if $y \notin \text{im}(\text{ver})$, then $\delta_{F'} \circ \text{ver} = 0$, then again by the congruence condition we have $\tilde{\mu}_{F'}(\cdot, \delta_{F'}^{(y)}) \equiv 0 \pmod{p}$. Therefore the corresponding summands in (i) and (ii) above vanishes modulo $\mathcal{T}(V)$.

Case(ii): y is not fixed by Δ : By Lemma 9.3, we have

$$\tilde{\mu}_{F'}(\cdot, \delta_{F'}^{(y)}) = \tilde{\mu}_{F'}(\cdot, \delta_{F'}^{(y^\gamma)}), \forall \gamma \in \Delta.
\tag{127}$$

Therefore the Δ -orbit of y in the sum is given by $\tilde{\mu}_{F'}(\cdot, \delta_{F'}^{(y)}) \sum_{\gamma \in \Delta} y^\gamma$, which belongs to $\mathcal{T}(V)$. \square

Viewing the elements $\int_{G_F} \epsilon \circ \text{ver}(x) d\mu_F(x)$ and $\int_{G_{F'}} \epsilon(x) d\tilde{\mu}_{F'}(x)$ in \mathbb{I} , as measures on the weight space \mathbb{W} , they are determined by their values on characters $\text{Hom}(\mathbb{W}, \overline{\mathbb{Q}}_p)$. Let $\nu := \int_{G_F} \epsilon \circ \text{ver}(x) d\mu_F(x) - \int_{G_{F'}} \epsilon(x) d\tilde{\mu}_{F'}(x)$. For any character $\chi : \mathbb{W} \rightarrow \overline{\mathbb{Q}}_p^\times$ let $\int_{\mathbb{W}} \chi(\gamma) d\nu(\gamma) \equiv 0 \pmod{p\mathbb{Z}_p}$. Then, it is easy to see that $\eta = \frac{1}{p}\nu$ defines a measure on \mathbb{W} , with $\eta(\chi) = \nu(\chi)/p$. If this happens, then $\nu \equiv 0 \pmod{p\mathbb{I}}$. We therefore have the following result.

Theorem 9.5. *The congruence*

$$\text{ver}(\mu_F) \equiv \tilde{\mu}_{F'} \pmod{\mathcal{T}}$$

hold if and only if

$$(128) \quad \int_{\mathbb{W}} \chi(y) \int_{G_F} \epsilon \circ \text{ver}(x) d\mu_F \equiv \int_{\mathbb{W}} \chi(y) \int_{G_{F'}} \epsilon(x) d\tilde{\mu}_{F'} \pmod{p\mathbb{Z}_p}$$

for all locally constant functions χ of \mathbb{W} , and for every locally constant \mathbb{Z}_p -valued function ϵ of $G_{F'}$ satisfying $\epsilon^\gamma = \epsilon$ for all $\gamma \in \Delta$.

We call the congruences that appear in equation (125) as torsion congruences over \mathbb{I} . The torsion congruences will be an important step towards proving the congruences in Theorem 7.40.

9.2 Remarks on Torsion Congruence in Families

We examine the torsion congruences in the case when the character ϵ of the group $G_{F'}$ is trivial. Recall that F is a totally real field and F' be a totally real extension of degree p . The Galois group $\Delta = \text{Gal}(F'/F)$ has order p . As before, f is a Hilbert modular form of weight $\kappa = (0, I)$ defined over F , and f' is the base-change of f to F' . We assume that both these modular forms are ordinary at all the primes above p . Then $\int_{G_F} d\mu_F(\sigma)$ is the p -adic L-function of f in \mathbb{I} , which we denote by $L_{p,F}$. These p -adic L-functions have been constructed in [Hid00b, §5.3.6] when $F = \mathbb{Q}$. A similar construction works over the totally real fields under some conditions [Ros15]. Note that there is no cyclotomic variable in these p -adic L-functions. Further, $\int_{G_{F'}} d\tilde{\mu}_{F'}(\sigma)$ is the image of the p -adic L-function of f' in \mathbb{I} . We denote this image by $\tilde{L}_{p,F'}$. These p -adic L-functions generate the characteristic ideals of the dual Selmer groups $\text{Sel}_F(\text{Ad}^0(\rho_f) \otimes \mathbb{I})$ and $\text{Sel}_{F'}(\text{Ad}^0(\rho_{f'}) \otimes \mathbb{I})$. The torsion congruence then takes following form

$$\int_{G_F} d\mu_F(\sigma) \equiv \int_{G_{F'}} d\tilde{\mu}_{F'}(\sigma) \pmod{p\mathbb{I}}.$$

This congruence can be written in the following way

$$(129) \quad L_{p,F} \equiv \tilde{L}_{p,F'} \pmod{p\mathbb{I}}.$$

The base-change morphism induces a map $\text{Sel}_F(\text{Ad}^0(\rho_f) \otimes \mathbb{I}) \rightarrow \text{Sel}_{F'}(\text{Ad}^0(\rho_{f'}) \otimes \mathbb{I})^\Delta$, which is an isomorphism. Since these Selmer groups have no non-trivial pseudonull submodules, we have an isomorphism $(\mathbb{I}/\tilde{L}_{p,F'})_\Delta \cong \mathbb{I}/L_{p,F}$. In particular, we have a surjection $\mathbb{I}/\tilde{L}_{p,F'} \rightarrow \mathbb{I}/L_{p,F}$. Hence, we have the equality: image of the ideal $(\tilde{L}_{p,F'}) = (L_{p,F})$ in \mathbb{I} . Therefore $\tilde{L}_{p,F'} \equiv uL_{p,F} \pmod{p\mathbb{I}}$ for some $u \in \mathbb{I}$. The congruences in the theorem above assert that a stronger congruence $\tilde{L}_{p,F'} \equiv L_{p,F} \pmod{p\mathbb{I}}$ might occur.

That this congruence might occur is suggested by the following observation. Consider the specialization of these p -adic L-functions to weight κ and κ' . Then for the above congruence (129), we need the congruence

$$(130) \quad L_{p,F}(P_\kappa) \equiv \tilde{L}_{p,F'}(P_{\kappa'}) \pmod{p\mathcal{O}}.$$

where P_κ and $P_{\kappa'}$ are the arithmetic points associated to f and f' respectively, and \mathcal{O} is a sufficiently large extension of \mathbb{Z}_p .

In the case when p divides the left hand side of this congruence, Hida showed in [Hid81], that the value $L_{p,F}(P_\kappa)$ is divisible by p . This theorem of Hida has been generalized by Ghate in [Gha02] to the case of totally real fields. Therefore, there is another Hilbert modular form g over F such that f is congruent to g modulo p . Such primes are known as congruence primes. The base-change of these Hilbert modular forms to the field F' are also congruent modulo p . In the case $F = \mathbb{Q}$, Hida also showed the converse, a congruence between modular forms modulo p , implies that p divides $L_{p,F}(P_\kappa)$. Therefore, an appropriate generalization of the theorem of Hida in *loc cit* (generalized by Ghate to the case of Hilbert modular forms over real quadratic fields) will show that p divides the value $\tilde{L}_{p,F'}(P_{\kappa'})$.

We hope to come back to the remaining cases and also the torsion congruences at a later time.

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